8.25.03 REVIEW - COMPLEX NUMBERS.

Appendix A

<u>Complex Number</u> $s = a + jb$ RECTANGULAR FORM

a, b are real
$j = \sqrt{-1}$

Real part of s : $Re\{s\} = a$
Imag part of s : $Im\{s\} = b$

$s = \rho e^{j\theta}$ POLAR FORM

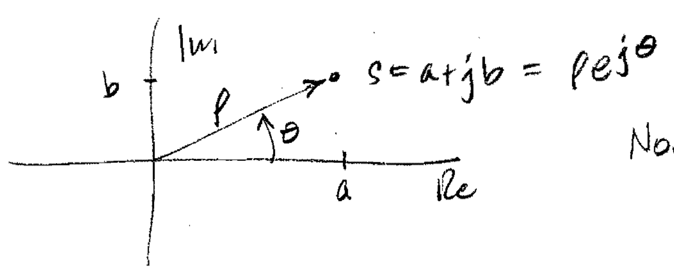
where $\rho \geq 0$ magnitude $\rightarrow$ $\rho = |s|$
θ angle (phase) $\rightarrow$ $\theta = \measuredangle s$

both ρ & θ are real numbers

Rectangular or polar form $\rightarrow$ Complex plane representation

$Re\{s\} = a$ $\rightarrow$ real axis (HORIZONTAL)
$Im\{s\} = b$ $\rightarrow$ imag axis (VERTICAL)

$s = a + jb$ ~ vector representation



Note $|s| = |\rho e^{j\theta}|$
$= |\rho| \, \cancel{|e^{j\theta}|}^{1} = |\rho| = \rho$
since $\rho \geq 0$

EQUALITY: $s_1 = s_2$ iff $Re\{s_1\} = Re\{s_2\}$ <u>AND</u> $Im\{s_1\} = Im\{s_2\}$

alternatively,
$s_1 = s_2$ iff $\rho_1 = \rho_2$ AND $\theta_1 = \theta_2$

Conversion b/w rectangular & polar coords

Euler's Formula $e^{j\theta} = \cos\theta + j\sin\theta$

$\Downarrow$ note

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$
$$= \cos\theta - j\sin\theta$$

$$\therefore \quad s = \rho e^{j\theta} = \rho(\cos\theta + j\sin\theta)$$
$$= a + jb$$

$\Downarrow$

$$\boxed{a = \rho\cos\theta \ \ \& \ \ b = \rho\sin\theta}$$

Furthermore, $a^2 = \rho^2\cos^2\theta$ & $b^2 = \rho^2\sin^2\theta$

since $\cos^2\theta + \sin^2\theta = 1 \rightarrow a^2+b^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta$
$$= \rho^2(\cos^2\theta + \sin^2\theta)$$
$$= \rho^2$$

$$\boxed{\therefore \ \rho = \sqrt{a^2+b^2}}$$

also $\dfrac{b}{a} = \dfrac{\rho\sin\theta}{\rho\cos\theta} = \tan\theta \rightarrow \boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$

in general,
$$\theta = \begin{cases} \tan^{-1}(b/a) & \text{when } a>0 \\ \tan^{-1}(b/a) + 180° & \text{when } a<0 \end{cases}$$

Complex conjugate of s $\rightarrow$ $\bar{s} = a - jb = \rho e^{-j\theta}$

Complex ADDITION $s_1 + s_2 = (a_1+a_2) + j(b_1+b_2)$

Polar domain preferred! $\Bigg\{$

Complex MULTIPLICATION $s_1 s_2 = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)$

Complex DIVISION $s_1/s_2 = \dfrac{(a_1a_2 + b_1b_2) + j(-a_1b_2 + a_2b_1)}{a^2+b^2}$

$\Uparrow$

perform as $\dfrac{s_1}{s_2} \times \dfrac{\bar{s}_2}{\bar{s}_2}$ ... $\nearrow$

Complex DIVISION (Polar Domain)

$$\frac{s_1}{s_2} = \frac{\rho_1 e^{j\theta_1}}{\rho_2 e^{j\theta_2}} = \frac{\rho_1}{\rho_2} e^{j(\theta_1-\theta_2)}$$

$\downarrow$

$$\left|\frac{s_1}{s_2}\right| = \frac{\rho_1}{\rho_2}$$

$$\measuredangle \frac{s_1}{s_2} = \theta_1 - \theta_2$$

MULTIPLICATION

$$s_1 s_2 = \rho_1 e^{j\theta_1} \cdot \rho_2 e^{j\theta_2} = \rho_1\rho_2 e^{j(\theta_1+\theta_2)}$$

$\downarrow$

$$|s_1 s_2| = \rho_1 \rho_2$$

$$\measuredangle s_1 s_2 = \theta_1 + \theta_2$$

Note: ADDITION / SUBTRACTION $\rightarrow$ Rectangular (Cartesian) Domain

MULTIPLICATION / DIVISION $\rightarrow$ Polar Domain

8·27·03

Addition of SINUSOIDS $x(t) = A_1 \cos(\omega t + \theta_1) + A_2 \cos(\omega t + \theta_2)$

same frequency
different amplitude & phase

Note: $\cos(\omega t + \theta_1) = Re\{e^{j(\omega t + \theta_1)}\}$

$$\therefore \quad x(t) = A_1 Re\{e^{j(\omega t+\theta_1)}\} + A_2 Re\{e^{j(\omega t+\theta_2)}\}$$

$$= Re\{A_1 e^{j(\omega t+\theta_1)} + A_2 e^{j(\omega t+\theta_2)}\}$$

$$= Re\{e^{j\omega t} \underbrace{[A_1 e^{j\theta_1} + A_2 e^{j\theta_2}]}_{\text{PHASOR}}\} = Re\{e^{j\omega t} \cdot A_3 e^{j\theta_3}\}$$

$$= A_3 Re\{e^{j(\omega t+\theta_3)}\}$$

$$= A_3 \cos(\omega t + \theta_3)$$

actual values of A_3 & θ_3 can be determined from A_1, A_2, θ_1 & θ_2

8·27·03 Ch 1 — Fundamental Concepts.

CONTINUOUS-TIME vs. DISCRETE-TIME

Signal. $x(t)$ real-valued (scalar-valued) fn of t.

some signals defy concise mathematical description
- could be represented by sample values
$\{x(t_0), x(t_1), \ldots, x(t_N)\}$

Manipulation of signals → Signal processing

Relationship b/w input and output → Systems.

Systems, Signals and Models can be represented & analyzed in time-domain or frequency domain

Continuous-Time Signals.

UNIT STEP FN. $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

Notes. $Ku(t) = \begin{cases} K & t \geq 0 \\ 0 & t < 0 \end{cases}$ and $x(t)u(t) = \begin{cases} x(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$

UNIT-RAMP FN. $r(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$

Note. $r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$

UNIT IMPULSE FN $\delta(t) = 0 \qquad t \neq 0$

$\int_{-\epsilon}^{\epsilon} \delta(\lambda)\, d\lambda = 1$ for any real $\epsilon > 0$

delta fn / Dirac distribution

Note. $\delta(t)$ is NOT defined at $t = 0$. also, $u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda$ all t, $t \neq 0$

PERIODIC SIGNALS $x(t+T) = x(t) \quad -\infty < t < \infty$

T ~ period (fixed, positive real number)

fundamental period is smallest value of T for which equation holds.

Note also periodic for integer multiples of T, i.e., $x(t+qT) = x(t)$

Sinusoids are periodic. $x(t) = A\cos(\omega t + \theta) \quad -\infty < t < \infty$

Note. $A\cos\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \theta\right] = A\cos(\omega t + 2\pi + \theta)$
$= A\cos(\omega t + \theta)$

$\therefore A\cos(\omega t + \theta)$ is periodic w/ period $T = \frac{2\pi}{\omega}$

$$x(t) = A\cos(\omega t + \theta)$$

$$x\left(t + \tfrac{2\pi}{\omega}\right) = A\cos\left(\omega\left(t + \tfrac{2\pi}{\omega}\right) + \theta\right) = A\cos(\omega t + 2\pi + \theta) = A\cos(\omega t + \theta) = x(t).$$

Note. $A\cos(\omega t - \pi/2) = A\sin(\omega t)$

skip

Is the sum of two periodic signals periodic?

if $x_1(t)$ is periodic w/ period T_1
& $x_2(t)$ is periodic w/ period T_2

then $x_1(t) + x_2(t)$ is periodic w/ period T
<u>iff</u> $\frac{T_1}{T_2} = \frac{q}{r}$ where q and r are integers.

in such a case, $T = rT_1 = qT_2$

TIME-SHIFTED SIGNAL

if t_1 is positive real number → $x(t-t_1)$ is $x(t)$ shifted RIGHT by t_1 seconds

→ $x(t+t_1)$ is $x(t)$ shifted LEFT by t_1 seconds.

use $u(t)$ to demonstrate.

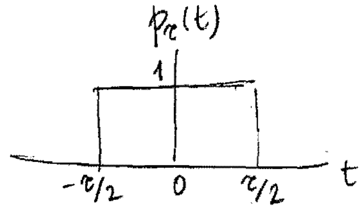
CONTINUOUS & PIECE-WISE CONTINUOUS SIGNALS

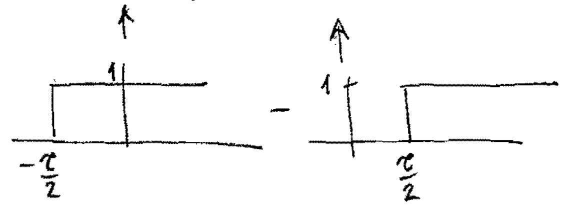
(discontinuous at fixed point t_1 if $x(t_1^-) \neq x(t_1^+)$)

(finite or countably infinite # of discontinuities. (ex. pulse train))

Note · rectangular pulse fn.

$$p_\tau(t) = u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)$$





SIGNALS DEFINED INTERVAL-BY-INTERVAL

$$x(t) = \begin{cases} x_1(t) & t_1 \le t < t_2 \\ x_2(t) & t_2 \le t < t_3 \\ x_3(t) & t \ge t_3 \end{cases}$$

→ interval defined by $u(t-t_1) - u(t-t_2)$

→ " $u(t-t_2) - u(t-t_3)$

→ " $u(t-t_3)$

$$\therefore\ x(t) = x_1(t)\left[u(t-t_1) - u(t-t_2)\right] + x_2(t)\left[u(t-t_2) - u(t-t_3)\right] + x_3 u(t-t_3)$$

$$= x_1 u(t-t_1) + \left[x_2(t) - x_1(t)\right] u(t-t_2) + \left[x_3(t) - x_2(t)\right] u(t-t_3)$$

DISCRETE-TIME SIGNALS. $x[n]$ $n \sim$ integer valued
↑ discrete points in time
$t = nT \to n$

SAMPLING (UNIFORM) $x[n] = x(t)\big|_{t=nT} = x(nT)$ $T \sim$ sampling interval (1/samp. freq.)

STEP FN. $u[n] = \begin{cases} 1 & n = 0, 1, 2, \ldots \\ 0 & n = -1, -2, \ldots \end{cases}$

RAMP FN. $r[n] = \begin{cases} n & n = 0, 1, 2, \ldots \\ 0 & n = -1, -2, \ldots \end{cases}$

$u[n]$ & $r[n]$ could be generated by sampling $u(t)$ & $r(t)$ w/ sampling interval, $T = 1$

UNIT PULSE $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

Note: unit pulse can <u>NOT</u> be generated by sampling $\delta(t)$

PERIODIC DISCRETE-TIME SIGNALS

$x[n+r] = x[n]$ $\forall$ integers n
↑ positive integer r

fundamental period - smallest value of r for which $x[n+r] = x[n]$

skip

Sinusoid - $x[n] = A\cos(\Omega n + \theta)$ $\Omega \sim$ discrete-time frequency

$x[n+r] = A\cos(\Omega n + \Omega r + \theta)$

for $x[n] = x[n+r]$, Ωr must be multiple of 2π
i.e.,

$\Omega r = 2\pi q$ for some integer q

$\Downarrow$

$x[n] = x[n+r] \longrightarrow$ if $\Omega = \frac{2\pi q}{r}$ for some positive <u>integers</u> q & r

Note. for $x[n] = A\cos(\Omega n + \theta)$

periodic iff $\Omega = 2\pi \frac{q}{r}$ where q & r are positive integers!

RECTANGULAR PULSE

$$p_L[n] = \begin{cases} 1 & n = -\frac{(L-1)}{2}, \ldots, -1, 0, 1, \ldots, \frac{(L-1)}{2} \\ 0 & \text{all other } n \end{cases}$$

Note. according to this definition, L should be odd.

DIGITAL SIGNAL (as opposed to discrete-time signal)

Let $\{a_1, a_2, \ldots, a_N\}$ be set of N real numbers

digital signal $x[n]$ is a discrete-time signal whose values belong to the finite set $\{a_0, a_1, \ldots, a_N\}$

DISCRETE-TIME SIGNAL — DISCRETE IN TIME
INFINITE POSSIBLE RANGE OF VALUES

DIGITAL SIGNAL — DISCRETE IN TIME
FINITE RANGE OF VALUES

Note: A sampled continuous-time signal is not necessarily a digital signal.

TIME SHIFTED SIGNAL

$x[n-q]$ q-step right shift of $x[n]$

$x[n+q]$ q-step left shift of $x[n]$

9.3.03 Ch 1 - Fundamental Concepts

DISCRETE-TIME SIGNALS DEFINED INTERVAL-BY-INTERVAL

$$x[n] = \begin{cases} x_1[n] & n_1 \le n < n_2 \\ x_2[n] & n_2 \le n < n_3 \\ x_3[n] & n \ge n_3 \end{cases}$$

interval $n_1 \to n_2 \to u[n-n_1] - u[n-n_2]$

" $n_2 \to n_3 \Rightarrow u[n-n_2] - u[n-n_3]$

" $n \ge n_3 \to u[n-n_3]$

$$x[n] = x_1[n]\left(u[n-n_1]-u[n-n_2]\right) + x_2[n]\left(u[n-n_2]-u[n-n_3]\right) + x_3[n]\,u[n-n_3]$$

$$= x_1[n]\,u[n-n_1] + \left(x_2[n]-x_1[n]\right)u[n-n_2] + \left(x_3[n]-x_2[n]\right)u[n-n_3]$$

READ § 1.4 - Examples of Systems.

BASIC SYSTEM PROPERTIES.

CAUSALITY (nonanticipatory)

if for any time t_1, the output response $y(t_1)$ at time t_1 resulting from input $x(t)$ does not depend on values of the input $x(t)$ for $t > t_1$

↑

current output not a fn of future input

or

no output prior to applied input

Note:

off-line or non real-time processing may be NONCAUSAL.

MEMORY vs. MEMORYLESS

output independent of past <u>input</u> values.

output response based on past <u>input</u> values

9·8·03 LINEARITY is $f(a_1x_1(t)) + f(a_2x_2(t)) = f(a_1x_1(t) + a_2x_2(t))$? test for LINEARITY

given inputs $x_1(t)$ & $x_2(t)$

w/corresponding outputs $y_1(t)$ & $y_2(t)$

and given scalars a_1 and a_2, system is <u>linear</u> iff.

$$\underbrace{a_1x_1(t) + a_2x_2(t)}_{\text{aggregate input}} \longrightarrow \underbrace{a_1y_1(t) + a_2y_2(t)}_{\text{aggregate output}}$$

aggregate input $\longrightarrow$ aggregate output.

(assuming no initial energy in system)

Note: Nonlinear systems are FREQUENTLY approximated by linear ones

TIME INVARIANCE is $y(t-t_1) = f(x(t-t_1))$? test for TIME-INVARIANCE

time shift in input $\longrightarrow$ time shift in output

i.e. if $x(t) \rightarrow y(t)$

and $x(t-t_1) \rightarrow y(t-t_1) \rightarrow$ TIME INVARIANT.

does $x(t) \rightarrow x(t-t_1)$ precipitate $y(t) \rightarrow y(t-t_1)$?

Likewise for discrete-time case, shift in input $x[n-n_1]$ yields corresponding shift in output $y[n-n_1]$

Eigenvalues of LTI systems are complex exponentials (&, thus, sinusoids)

LINEAR I/O DIFF EQ w/ CONSTANT COEFFICIENTS.
↑ continuous-time

$$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{M} b_i x^{(i)}(t)$$

where $y^{(i)}(t) = \frac{d^i}{dt^i}[y(t)]$

$\{a_0, a_1, \ldots, a_{N-1}, b_0, b_1, \ldots, b_M\} \in \mathbb{R}$ constants.

From § 1.5 p.45
→ Diff. Eq. w/constant coefficients ⇒ LTI

To solve Diff Eq, need initial conditions for derivatives,
i.e.,
$y(0), y^{(1)}(0), \ldots, y^{(N-1)}(0)$ or at time 0^-

1st ORDER DIFF EQ// $\frac{dy(t)}{dt} + ay(t) = bx(t)$

$$\Downarrow$$

Solution// $$\boxed{y(t) = e^{-at}y(0) + \int_0^t e^{-a(t-\lambda)} bx(\lambda)\,d\lambda \quad t \geq 0}$$

Note: or time 0^- & need to know $y(0)$
assuming $x(t)$ applied for $t \geq 0$

FYI, if given $\frac{dy(t)}{dt} + ay(t) = b_1 \frac{dx(t)}{dt} + b_0 x(t)$

$$\Downarrow$$

Solution// $$y(t) = e^{-at}[y(0) - b_1 x(0)] + \int_0^t e^{-a(t-\lambda)}(b_0 - ab_1)\, x(\lambda)d\lambda + b_1 x(t) \quad t \geq 0$$

Note: or time 0^-
& need $y(0)$ & $x(0)$

Ex. / 2.1 (a) $\underset{a=-2}{\frac{dy(t)}{dt} - 2y(t)} = \underset{b=1}{x(t)}$ $\quad x(t) = u(t)$, $y(0) = 1$

$$y(t) = e^{-(-2)t}\cancel{y(0)}^{1} + \int_0^t e^{-(-2)(t-\lambda)}(1)\cancel{x(\lambda)}^{1} d\lambda \quad t \geq 0$$

$$= e^{2t} + e^{2t}\int_0^t e^{-2\lambda}\, d\lambda$$

$$= e^{2t} + e^{2t}\left(\frac{1}{-2}\right)\left[e^{-2t} - 1\right]$$

$$= e^{2t} + \frac{1}{2}e^{2t} - \frac{1}{2}$$

$$= \frac{3}{2}e^{2t} - \frac{1}{2} \qquad t \geq 0$$

Ex. / 2.1 (d) $\underset{a=10}{\frac{dy(t)}{dt} + 10y(t)} = \underset{b=2}{2x(t)}$ $\quad x(t) = e^{4t}u(t)$, $y(0) = 0$

$$y(t) = e^{-10t}\cancel{y(0)}^{0} + \int_0^t e^{-10(t-\lambda)}(2)\cancel{x(\lambda)}^{e^{4\lambda}} d\lambda$$

$$= 2e^{-10t}\int_0^t e^{10\lambda} \cdot e^{4\lambda}\, d\lambda$$

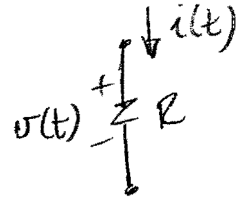
$$= 2e^{-10t}\int_0^t e^{14\lambda}\, d\lambda$$

$$= 2e^{-10t} \cdot \frac{1}{14}\left[e^{14t} - 1\right]$$

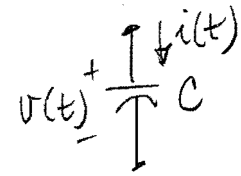
$$= \frac{1}{7}e^{4t} - \frac{1}{7}e^{-10t} \quad t \geq 0$$

SYSTEM MODELING - ELECTRIC CKTS.

Resistor $v(t) = R\,i(t)$

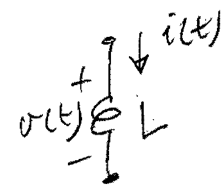


Capacitor $\dfrac{dv(t)}{dt} = \dfrac{1}{C} i(t)$



& $v(t) = \dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i(\lambda)\,d\lambda$

Inductor $v(t) = L\dfrac{di(t)}{dt}$

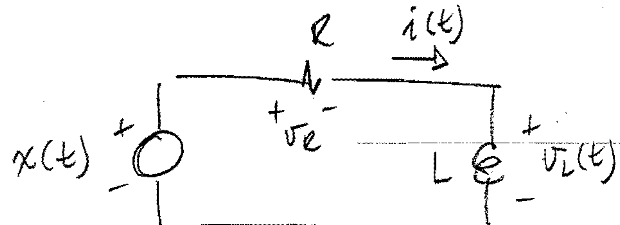


& $i(t) = \dfrac{1}{L}\displaystyle\int_{-\infty}^{t} v(\lambda)\,d\lambda$

KVL → Σ voltage drops = Σ adds.

KCL → Σ i into node = 0.

EX. 2.8



I/O Eq b/w $x(t)$ & $i(t)$

$$\left.\begin{aligned} v_R(t) &= R\,i(t) \\ v_L(t) &= L\frac{di(t)}{dt} \end{aligned}\right\} \quad x(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$\Rightarrow \frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{1}{L} x(t)$$

$$\left.\begin{aligned} x(t) &= u(t) \\ a = \tfrac{R}{L}&,\ b = \tfrac{1}{L} \end{aligned}\right\} \rightarrow i(t) =$$

$$i(t) = e^{-\frac{R}{L}t} i(0) + \int_0^t e^{-\frac{R}{L}(t-\lambda)} \left(\frac{1}{L}\right) d\lambda$$

EX. / 2.8 (cont'd)

$$i(t) = e^{-\frac{R}{L}t} i(0) + \frac{1}{L} e^{-\frac{R}{L}t} \int_0^t e^{\frac{R\lambda}{L}} d\lambda$$

$$= e^{-\frac{R}{L}t} i(0) + \frac{1}{L} e^{-\frac{R}{L}t} \cdot \frac{L}{R} \left[e^{\frac{R t}{L}} - 1 \right]$$

$$= e^{-\frac{R}{L}t} i(0) + \frac{1}{R} \left[1 - e^{-\frac{R}{L}t} \right] \quad t \geq 0$$

$$v_L(t) = L \frac{di(t)}{dt}$$

$$= L \frac{d}{dt} \left[\qquad \right]$$

$$= L \cdot \left(-\frac{R}{L}\right) e^{-\frac{R}{L}t} i(0) + L \cdot \frac{1}{R} \cdot -\left(-\frac{R}{L}\right) e^{-\frac{R}{L}t}$$

$$= -R e^{-\frac{R}{L}t} i(0) + e^{-\frac{R}{L}t} \quad t \geq 0$$

if $i(0) = 0$

$$\rightarrow v_L(t) = e^{-\frac{R}{L}t} \quad t \geq 0$$

ALTERNATIVELY

(a) I/O DIFF EQ b/w $v_L(t)$ & $x(t)$

$v_L(t) = L \frac{di(t)}{dt}$ & $v_R(t) = R i(t)$

$\Downarrow$

$$x(t) = R i(t) + \underbrace{L \frac{di(t)}{dt}}_{v_L(t)} \xrightarrow{d/dt} \frac{dx(t)}{dt} = \underbrace{R \frac{di(t)}{dt}}_{= \frac{R}{L} v_L(t)} + \frac{dv_L(t)}{dt}$$

$$\therefore \frac{dv_L(t)}{dt} + \frac{R}{L} v_L(t) = \frac{dx(t)}{dt}$$

(b)

if $x(t) = u(t) \rightarrow \frac{dx(t)}{dt} = \delta(t) \Rightarrow \frac{dv_L(t)}{dt} + \frac{R}{L} v_L(t) = \delta(t)$

OVER →

$$\therefore \quad v_L(t) = e^{-\frac{R}{L}t} v_L(0) + \int_0^t e^{-\frac{R}{L}(t-\lambda)} (1)\,\delta(\lambda)\, d\lambda$$

$$= e^{-\frac{R}{L}t} v_L(0) + e^{-\frac{R}{L}t} \cdot \left[\cancel{e^{-\frac{R}{L}(-0)}}^{\,1}\right]$$

$$= e^{-\frac{R}{L}t} v_L(0) + e^{-\frac{R}{L}t} \qquad t \geq 0$$

if $v_L(0) = 0 \rightarrow \quad v_L(t) = e^{-\frac{R}{L}t} \quad t \geq 0$ same as previous answer.

(c) $v_L(0) = v_0$ & $x(t) = 1 \quad t \geq 0$

$$v_L(t) = e^{-\frac{R}{L}t} \cdot [1 + v_0] \qquad t \geq 0$$

(d) $v_L(0) = 0$ & $x(t) = 1 + \frac{R}{L}t \quad t \geq 0$

$$\frac{dx(t)}{dt} = \delta(t) + \frac{R}{L} u(t)$$

$$\therefore \quad v_L(t) = e^{-\frac{R}{L}t} \cancel{v_L(0)}^{\,0} + \int_0^t e^{-\frac{R}{L}(t-\lambda)} \left[1\,\delta(\lambda) + \frac{R}{L} u(\lambda)\right] d\lambda$$

$$= e^{-\frac{R}{L}t} + \frac{\cancel{R}}{\cancel{L}}\left(-\frac{\cancel{L}}{\cancel{R}}\right) e^{-\frac{R}{L}t}\left[e^{-\frac{R}{L}(-t)} - 1\right]$$

$$= e^{-\frac{R}{L}t} - e^{-\frac{R}{L}t} + 1$$

$$v_L(t) = 1 \qquad t \geq 0$$

9-10-03 Ch 2 - System Definition

LINEAR I/O DIFFERENCE EQ w/ CONSTANT COEFFICIENTS

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-i]$$

where $\{a_1, a_2, \ldots, a_N, b_0, b_1, \ldots, b_M\} \in \mathbb{R}$ constants

Note: from §1.5 p.45, constant coefficients $\Rightarrow$ $\cancel{L}$TI

if $y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i]$

has any $a_i \neq 0$ $\rightarrow$ Recursive sol'n (requires N previous values $\{y[n-1], \ldots, y[n-N]\}$)

if all $a_i = 0$ for $i = 1, \ldots, N$ $\rightarrow$ Nonrecursive

$\Downarrow$

$$y[n] = \sum_{i=0}^{M} b_i x[n-i]$$

9-15-03

FIRST ORDER LINEAR DIFFERENCE EQUATION

$$y[n] = -ay[n-1] + bx[n] \quad n = 1, 2, \ldots$$

w/ initial condition $y[0]$

$\Downarrow$

$n=1$ $\quad y[1] = -ay[0] + bx[1]$

$n=2$ $\quad y[2] = -ay[1] + bx[2]$

$$= -a(-ay[0] + bx[1]) + bx[2]$$

$$= a^2 y[0] - abx[1] + bx[2]$$

9.15.03 Ch 2 - System Definition

$n=3 \quad y[3] = -a y[2] + b x[3]$

$= -a\left(a^2 y[0] - ab\,x[1] + b\,x[2]\right) + b\,x[3]$

$= \underset{\uparrow\atop (-a)^n}{-a^3 y[0]} + \underset{\uparrow\atop (-a)^{n-1}b}{a^2 b\,x[1]} - \underset{\uparrow\atop (-a)^{n-2}b}{ab\,x[2]} + \underset{\uparrow\atop (-a)^{n-3}b}{b\,x[3]}$

$\therefore \quad y[n] = (-a)^n y[0] + \sum_{i=1}^{n} (-a)^{n-i} b\,x[i]$

2.19/ (b) closed-form sol'n for $y[n]$ w/ $x[n] = u[n]$, $y[0] = 0$

(i) $y[n+1] + 1.5y[n] = x[n]$

$n=0 \quad y[1] = -1.5\cancel{y[0]}^{0} + \cancel{x[0]}^{1}$
$= 1$

$n=1 \quad y[2] = -1.5\,y[1] + x[1]$

$= -1.5(1) + 1$

$n=2 \quad y[3] = -1.5\,y[2] + x[2]$

$= -1.5\left(-1.5\cdot y[1] + 1\right) + 1$

$= (-1.5)^2 y[1] - 1.5 + 1$

$n=3 \quad y[4] = -1.5\left[(-1.5)^2 y[1] - 1.5 + 1\right] + 1$

$= (-1.5)^3 \cancel{y[1]}^{1} + (-1.5)^2 + (-1.5) + 1$

$= (-1.5)^{n-1} + (-1.5)^{n-2} + (-1.5)^{n-3} + (-1.5)^{n-4}$

$= \sum_{i=1}^{n} (-1.5)^{n-i}$

9.15.03 Ch 2 - System Definition

2.19/

$$y[n] = \sum_{i=1}^{n} (-1.5)^{n-i}$$

Note. $\sum_{i=1}^{N} a^i = \frac{a - a^{N+1}}{1-a}$

FYI - $\sum_{i=0}^{N} a^i = \frac{1-a^{N+1}}{1-a}$

$$y[n] = (-1.5)^n \cdot \sum_{i=1}^{n} (-1.5)^{-i}$$

$$= (-1.5)^n \cdot \sum_{i=1}^{n} \left(\frac{-1}{1.5}\right)^i$$

$$= (-1.5)^n \cdot \frac{(-1/1.5) - (-1/1.5)^{n+1}}{1 - (-1/1.5)}$$

$$= \frac{(-1.5)^{n-1} - (-1.5)^n \cdot (-1/1.5)^{n+1}}{(2.5/1.5)}$$

$$= \frac{1}{2.5} \cdot (1.5)(-1.5)^{n-1} - \frac{1}{2.5} \cdot (1.5) \cdot (-1.5)^n \cdot (-1.5)^{-(n+1)}$$

$$= 0.4 \frac{(1.5) \cdot (-1.5)^n}{-1.5} - 0.4 \cdot (1.5) \cdot (-1.5)^{-1}$$

$$= -0.4 (-1.5)^n + 0.4 \quad n \geq 0$$

(c) closed-form $y[n]$ w/ $x[n] = u[n]$, $y[0] = 2$

$n=0$ $\quad y[1] = -1.5\, y[0] + x[0]$

$n=1$ $\quad y[2] = -1.5\, y[1] + x[1]$

$$= -1.5 \left[-1.5 y[0] + x[0]\right] + x[1]$$

$$= (-1.5)^2 y[0] - 1.5 x[0] + x[1]$$

9.15.03 Ch 2 System Definition

2.19/ $n=2$ $y[3] = -1.5y[2] + x[2]$

$$= (-1.5)^3 y[0] + (-1.5)^2 x[0] + (-1.5)^1 x[1] + (-1.5)^0 x[2]$$

$$\therefore\ y[n] = (-1.5)^n \cancel{y[0]}^{2} + \sum_{i=1}^{n} (-1.5)^{n-i} \cancel{x[i-1]}^{1} \qquad i \geq 1$$

$$= 2 \cdot (-1.5)^n + \underbrace{\sum_{i=1}^{n} (-1.5)^{n-i}}_{\text{can use previous answer.}}$$

9.15.03 Ch 3 Convolution Representation.

DISCRETE TIME LTI SYSTEMS

Note: - Assume $y[n]$ is output response from $x[n]$ (input) w/ <u>NO</u> initial energy prior to application of $x[n]$!

UNIT-PULSE RESPONSE

if input $x[n] = \delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

↓ then

output $y[n] = h[n]$ ~ unit-pulse response

Given system response to single pulse, if system is causal LTI then output response $y[n]$ can be computed for any arbitrary input $x[n]$

Assume input $x[n] = 0$ for $n = -1, -2, \ldots$

for any shift in time i, $\delta[n-i] = \begin{cases} 1 & n=i \\ 0 & n \neq i \end{cases}$

$$\therefore\ x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

$$= \sum_{i=0}^{\infty} x[i]\delta[n-i] \qquad n = 0, 1, 2, \ldots$$

9.15.03 Ch 3 Convolution Representation

Since system is LTI, system output response to input $x[i]\,\delta[n-i]$

$$\Downarrow$$

is $y_i[n] = x[i]\,h[n-i]$

complete solution is thus

$$y[n] = \sum_{i=0}^{\infty} y_i[n] = \sum_{i=0}^{\infty} x[i]\,h[n-i] \qquad n \geq 0 \quad \text{(CAUSAL)}$$

$\Uparrow$

CONVOLUTION of $x[n]$ & $h[n]$

$$\sum_{i=0}^{\infty} x[i]\,h[n-i] = \underbrace{x[n]}_{\text{input}} * \underbrace{h[n]}_{\text{unit-pulse response}} = \underbrace{y[n]}_{\text{output}} \qquad n \geq 0$$

In essence, system is completely determined by $h[n]$

$\therefore$ if $h[n]$ is known, output response $y[n]$ from any arbitrary $x[n]$ can be determined via

$$y[n] = x[n] * h[n]$$

9.17.03

DISCRETE-TIME CONVOLUTION

in general $\quad x[n] * v[n] = \sum_{i=-\infty}^{\infty} x[i]\,v[n-i]$ (CONVOLUTION SUM)

if $x[n] = 0$ for $n < 0$ & $v[n] = 0$ for $n < 0$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$x[i] = 0 \quad i < 0 \qquad\qquad v[n-i] = 0 \quad n-i < 0 \quad (n < i)$

$$\therefore\ x[n] * v[n] = \begin{cases} 0 & n = -1, -2, \ldots \\ \sum_{i=0}^{n} x[i]\,v[n-i] & n = 0, 1, 2, \ldots \end{cases}$$

9.17.03 Ch3 Convolution Representation

Convolution process · convolving $x[n]$ & $v[n]$

1/ change $n \to i$ $\therefore$ $x[n] \to x[i]$
$v[n] \to v[i]$

2/ create $v[n-i]$ by

a/ fold $v[i]$ about $i=0$ axis $\to v[-i]$
b/ time shift $v[-i]$ by $n \to v[n-i]$

3/ once $x[i]$ & $v[n-i]$ generated, sum elements

$$\sum_{i=-\infty}^{\infty} x[i]\, v[n-i]$$

Emphasize value of text examples 3.2 & 3.3.

graphical ↑ analytical ↑

PROPERTIES OF CONVOLUTION OPERATION $x[n] * v[n]$

Associativity – $x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$

Commutativity – $x[n] * v[n] = v[n] * x[n]$

$$\Downarrow$$

$$\sum_{i=-\infty}^{\infty} x[i]\, v[n-i] = \sum_{i=-\infty}^{\infty} v[i]\, x[n-i]$$

Distributivity w/ addition –

$$x[n] * (v[n] + w[n]) = x[n] * v[n] + x[n] * w[n]$$

Convolution w/ unit pulse – $x[n] * \delta[n] = x[n]$

Convolution w/ shifted unit pulse – $x[n] * \delta[n-q] = x[n-q]$

9.17.03 Ch3 Convolution Representation

Causal LTI System

$$y[n] = x[n] * h[n] = \sum_{i=0}^{\infty} x[i]\, h[n-i] \qquad n \geq 0$$

$\uparrow x[n] = 0,\ n < 0$

Since $h[n] = 0$ $n<0$ (causality) $\rightarrow h[n-i] = 0$ $i > n$

$$\therefore\ y[n] = \sum_{i=0}^{n} x[i]\, h[n-i] \qquad n \geq 0$$

$$= \sum_{i=0}^{n} h[i]\, x[n-i] \qquad n \geq 0$$

since $x[n] * h[n] = h[n] * x[n]$

skip.

If system is non-causal $\rightarrow h[n] \neq 0$ for $n < 0$

$$\Downarrow$$

$$y[n] = \sum_{i=0}^{\infty} x[i]\, h[n-i]$$

if $x[n] \neq 0,\ n<0$ and noncausal system

$$\Downarrow$$

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

9.22.03

CONTINUOUS-TIME LTI SYSTEMS.

Note: Assume $y(t)$ is response from $x(t)$ (input) w/ no initial energy in system prior to application of $x(t)$

IMPULSE RESPONSE if $x(t) = \delta(t) \longrightarrow y(t) = h(t)$

impulse response

remember

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\epsilon}^{\epsilon} \delta(t)\, dt = 1 \qquad \text{for any } \epsilon > 0.$$

9.22.03 Ch 3 Convolution Representation

As in discrete-time case, $x(t) = \delta(t) \longrightarrow h(t)$

$$x(t) = \delta(t-\lambda) \longrightarrow h(t-\lambda)$$

if $x(t)$ is arbitrary input w/ $x(t) = 0 \quad t<0$

then
$$x(t) = \int_{0^-}^{\infty} x(\lambda)\,\delta(t-\lambda)\,d\lambda \qquad t \geq 0.$$

given input $x(\lambda)\delta(t-\lambda)$ and $\delta(t-\lambda) \rightarrow h(t-\lambda)$

$\downarrow$

output $y_\lambda(t) = x(\lambda)h(t-\lambda)$

integrating over all λ gives
$$y(t) = \int_{0^-}^{\infty} x(\lambda)\,h(t-\lambda)\,d\lambda \qquad t \geq 0.$$

In general, $x(\lambda)h(t-\lambda)$ does not contain impulse @ $\lambda = 0$.
As such,
$$y(t) = \int_{0}^{\infty} x(\lambda)\,h(t-\lambda)\,d\lambda \qquad t \geq 0$$
$$= x(t) * h(t) \qquad t \geq 0$$

As in discrete-time case, system is completely determined by $h(t)$ in that if $h(t)$ is known, output response to any arbitrary input can be computed.

CONTINUOUS-TIME CONVOLUTION.

$$x(t) * v(t) = \int_{-\infty}^{\infty} x(\lambda)\,v(t-\lambda)\,d\lambda \qquad \text{(CONVOLUTION INTEGRAL)}$$

$$\left.\begin{array}{l}\text{if } x(t) = 0 \quad t<0 \\ \& \ v(t) = 0 \quad t<0\end{array}\right\} \rightarrow x(t) * v(t) = \int_{0}^{t} x(\lambda)\,h(t-\lambda)\,d\lambda \qquad t \geq 0$$

9.22.03 Ch 3 - Convolution Representation.

Continuous Time Convolution Process.

1/ Graph $x(\lambda)$ & $v(-\lambda)$ as fn of λ

2/ Define interval $[0,a]$

largest value of a for which $x(\lambda)v(t-\lambda)$ has same analytical form for all $t \in [0,a]$

graph $v(t-\lambda)$ & $x(\lambda)v(t-\lambda)$

3/ w/ $0 \le t \le a \rightarrow \int_{\lambda=0}^{t} x(\lambda)v(t-\lambda)\,d\lambda$

$= x(t) * v(t)$ over $t \in [0,a]$

4/ Define interval $[a,b]$

largest value of b for which $x(\lambda)v(t-\lambda)$ has same analytical form for all $t \in [a,b]$

graph $v(t-\lambda)$ & $x(\lambda)v(t-\lambda)$

5/ w/ $a \le t \le b \rightarrow \int_{\lambda=0}^{t} x(\lambda)v(t-\lambda)\,d\lambda$

$= x(t) * v(t)$ over $t \in [a,b]$

6/ Repeat steps 4 & 5 until $x(t) * v(t)$ is computed for all $t > 0$.

Encourage students to review Examples ~~3.7~~ 2.15 & ~~3.8~~ 2.16 in text.

9.22.03 Ch 3 - Convolution Representation

PROPERTIES OF CONVOLUTION

ASSOCIATIVITY $[x(t)*v(t)]*w(t) = x(t)*[v(t)*w(t)]$

COMMUTATIVITY $x(t)*v(t) = v(t)*x(t)$

$$\therefore \int_{-\infty}^{\infty} x(\lambda)\,v(t-\lambda)\,d\lambda = \int_{-\infty}^{\infty} v(\lambda)\,x(t-\lambda)\,d\lambda$$

DISTRIBUTIVITY w/ADDITION $x(t)*[v(t)+w(t)] = x(t)*v(t) + x(t)*w(t)$

DERIVATIVE PROPERTY
$$\frac{d}{dt}[x(t)*v(t)] = \dot{x}(t)*v(t)$$
$$= \dot{v}(t)*x(t)$$
$$= \frac{d}{dt}[v(t)*x(t)]$$

CONVOLUTION w/ UNIT IMPULSE

$$x(t)*\delta(t) = \delta(t)*x(t) = \int_{-\infty}^{\infty}\delta(\lambda)\,x(t-\lambda)\,d\lambda$$
$$= \int_{-\infty}^{\infty}\delta(\lambda)\,x(t)\,d\lambda$$
$$= x(t)\int_{-\infty}^{\infty}\delta(\lambda)\,d\lambda$$
$$= x(t)$$

CONVOLUTION w/ SHIFTED UNIT IMPULSE

if $\delta_c(t) = \delta(t-c) \rightarrow x(t)*\delta(t-c) = x(t-c)$

Causal LTI System

$$y(t) = x(t)*h(t) = \int_{0}^{t} x(\lambda)\,h(t-\lambda)\,d\lambda \qquad t \geq 0$$

$h(t)=0 \quad t<0$ $\qquad x(t)=0,\ t<0$

by causality $\rightarrow$ upper limit on $\int^t$ is t.

9·22·03 Ch 3 - Convolution Representation

also, $y(t) = h(t) * x(t) = \int_0^t h(\lambda)\, x(t-\lambda)\, d\lambda \quad t \geq 0.$

Going back to $y(t) = x(t) * h(t) = \int_0^t x(\lambda)\, h(t-\lambda)\, d\lambda \quad t \geq 0$

skip.

if system is non-causal $\rightarrow$ $h(t) \neq 0$ for $t<0$

$$\Downarrow$$

$$y(t) = \int_0^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

if $x(t) \neq 0$ for $t<0$ and system is noncausal,

$$\Downarrow$$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

9·24·03 Ch 4 - Fourier Series / Fourier Transform

$x(t)$ ~ arbitrary signal

usually expressed/generated in time domain

signals w/ sharp (temporal) transitions are well-characterized in time domain } impulse, pulse pulse trains.

some signals have features that are better characterized in other domains
→ frequency
→ wavelet
→ transform domain

Fourier analysis → Fourier domain = Frequency domain

9.24.03 Ch 4 Fourier Series | Fourier Transform

Frequency Content
- generated by decomposing $x(t)$ into constituent sinusoids.

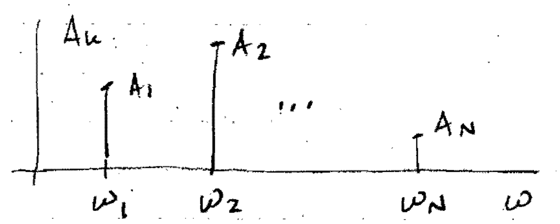
In general, suppose $x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \theta_k)$ $\quad -\infty < t < \infty$

$$\left\{\begin{array}{ll} \{A_1, A_2, \ldots, A_N\} & \text{are amplitudes (nonnegative)} \\ \{\omega_1, \omega_2, \ldots, \omega_N\} & \text{are frequencies (rad/sec)} \\ \{\theta_1, \theta_2, \ldots, \theta_N\} & \text{are phases (radians)} \end{array}\right.$$

together, completely describe $x(t)$

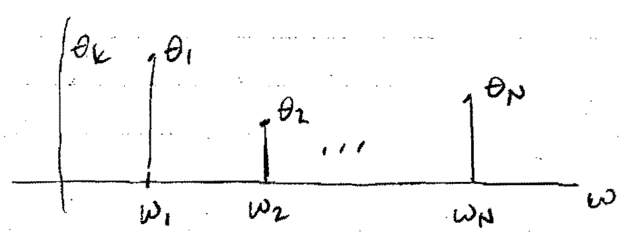
— Plotting A_k vs. ω_k

AMPLITUDE SPECTRUM
(LINE SPECTRUM)



— Plotting θ_k vs. ω_k

PHASE SPECTRUM
(LINE SPECTRUM)



Euler's Formula

$$A_k e^{j(\omega_k t + \theta_k)} = A_k \cos(\omega_k t + \theta_k) + j A_k \sin(\omega_k t + \theta_k)$$

$$\therefore \quad A_k \cos(\omega_k t + \theta_k) = \text{Re}\left\{A_k e^{j(\omega_k t + \theta_k)}\right\}$$

$$\Downarrow$$

$$x(t) = \sum_{k=1}^{N} \text{Re}\left\{A_k e^{j(\omega_k t + \theta_k)}\right\}$$

9.24.03 Ch 4 - Fourier Series / Fourier Transform

Furthermore,

$$Re\{A_k e^{j(\omega_k t+\theta_k)}\} = \frac{A_k}{2} e^{j(\omega_k t+\theta_k)} + \frac{A_k}{2} e^{-j(\omega_k t+\theta_k)}$$

$$\therefore \; x(t) = \sum_{k=1}^{N} \left[\frac{A_k}{2} e^{j(\omega_k t+\theta_k)} + \frac{A_k}{2} e^{-j(\omega_k t+\theta_k)} \right]$$

if we let $c_k = \frac{A_k}{2} e^{j\theta_k}$ & $c_{-k} = \frac{A_k}{2} e^{-j\theta_k}$

then

$$x(t) = \sum_{k=1}^{N} \left[c_k e^{j\omega_k t} + c_{-k} e^{-j\omega_k t} \right]$$

$$= \sum_{k=1}^{N} c_k e^{j\omega_k t} + \sum_{k=1}^{N} c_{-k} e^{j(-\omega_k)t}$$

"negative" frequency
- mathematical abstraction
- no physical meaning

$$x(t) = \sum_{\substack{k=-N \\ k\neq 0}}^{N} c_k e^{j\omega_k t}$$

$$c_k = \frac{A_k}{2} [\cos\theta_k + j\sin\theta_k] \quad k=1,2,\dots,N$$

$$c_{-k} = \frac{A_k}{2} [\cos\theta_k - j\sin\theta_k] \quad k=1,2,\dots,N$$

Note: c_k & c_{-k} are real iff $\sin\theta_k = 0$
$\Downarrow$
$\theta_k = n\pi$
n is integer

Also, $|c_k| = \frac{A_k}{2} = |c_{-k}| \quad k=1,2,\dots$

magnitude → Amplitude spectrum is <u>even</u> fn of ω

$\measuredangle c_k = -\measuredangle c_{-k} \quad k=1,2,\dots$

phase → phase spectrum is <u>odd</u> fn of ω.

9.24.03 Ch 4 Fourier Series / Fourier Transform.

In general, $x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \theta_k) \quad -\infty < t < \infty$

limited # of signals can be represented by this expression

if $N \to \infty$, then

$$x(t) = \sum_{k=1}^{\infty} A_k \cos(\omega_k t + \theta_k) \quad -\infty < t < \infty$$

set of signals includes periodic signals.

9.29.03

FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS.

Periodic signals $\quad x(t+T) = x(t) \quad \forall t \quad -\infty < t < \infty$

fundamental period T

[smallest Δt for which $x(t+T) = x(t)$]

if $x(t)$ is PERIODIC w/ period T, $x(t)$ can be expressed as sum of complex exponentials

i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad -\infty < t < \infty$$

Fourier series of periodic signal $x(t)$

$\omega_0 = \frac{2\pi}{T}$ rad/sec

fundamental frequency

Determining $c_k \longrightarrow$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt \quad k = 0, \pm 1, \pm 2, \ldots$$

integral valid over any full period s.t. $c_k = \frac{1}{T}\int_{0}^{T} x(t) e^{-jk\omega_0 t}\, dt \quad k = 0, \pm 1, \pm 2, \ldots$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$$

9.29.03 Ch 4 Fourier Series / Fourier Transform

For $k=0 \rightarrow$ dc component $x(t)\cancel{e^{j0\cdot\omega_0 t}} = x(t)$

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

In general, a <u>periodic</u> signal $x(t)$ has a Fourier Series if it satisfies <u>DIRICHLET CONDITIONS</u>

1/ $x(t)$ is absolutely integrable over any period

$$\int_a^{a+T} |x(t)|\,dt < \infty \quad \text{for any } a$$

2/ $x(t)$ finite number of maxima/minima over any period

3/ $x(t)$ finite number of discontinuities over any period.

Gibbs Phenomenon - Recitation MATLAB p. 161

Parseval's Theorem

$x(t)$ ~ periodic w/ period T

average power $\rightarrow$ $P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$ integrate over time

$= \sum_{k=-\infty}^{\infty} |c_k|^2$ sum over line spectra.

9.29.03 Ch4 Fourier Series / Fourier Transform

FOURIER TRANSFORM

periodic signals → Fourier Series

What about nonperiodic (aperiodic) signals?

⇓

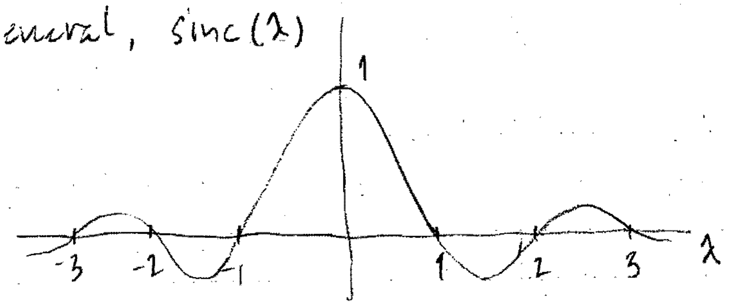
Fourier Transform (continuous ω)

Note: book goes through derivation from $k\omega_0 \to \omega$ (as $T \to \infty$)

in so doing, uses sinc(·) fn.

$$\text{sinc}\,\lambda = \frac{\sin(\pi\lambda)}{\pi\lambda}$$

in general, sinc(λ)



zero crossings @ $\lambda = \pm 1, \pm 2, \ldots$

sinc(0) = 1 w/ subsequent sidelobes of decreasing magnitude

10.01.03

FOURIER TRANSFORM

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad -\infty < \omega < \infty$$

↑ ω is continuous frequency

cf. Fourier Series w/ $c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}$ $\quad k = 0, \pm 1, \pm 2, \ldots$

harmonics of fundamental freq. ($k\omega_0$) ($\omega_0 = \frac{2\pi}{T}$)

10.01.03 Ch 4 Fourier Series / Fourier Transform

Inverse Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$$\therefore \quad x(t) \underset{IFT}{\overset{FT}{\rightleftarrows}} X(\omega)$$

Fourier Transform exists if $x(t)$ is absolutely integrable

$$\Downarrow$$

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

$$\Downarrow$$

"well-behaved" – finite # of discontinuities, maxima, minima w/in any finite Δt

All actual signals are well-behaved and, thus, absolutely integrable.

$$\underbrace{X(\omega)}_{\text{Complex quantity}} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

(even though ω is REAL)

$$\Downarrow$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\,dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,dt$$

$$= R(\omega) + jI(\omega)$$

where $R(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\,dt$

and $I(\omega) = -\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,dt$

Polar form $\rightarrow$ $X(\omega) = |X(\omega)|\, e^{j\measuredangle X(\omega)}$

where $|X(\omega)| = (R^2(\omega) + I^2(\omega))^{1/2}$

$\measuredangle X(\omega) = \arctan\left(I(\omega)/R(\omega)\right)$

10.01.03 Ch 4 Fourier Series | Fourier Transform

Note: If $x(t)$ is REAL valued, $X(-\omega) = \overline{X(\omega)} = |X(\omega)| e^{-j\angle X(\omega)}$

$\downarrow$

$\therefore \quad |X(-\omega)| = |X(\omega)|$

$\angle X(-\omega) = -\angle X(\omega)$

SIGNALS w/ SYMMETRY . EVEN OR ODD.

$x(t) = x(-t)$ $\qquad$ $x(-t) = -x(t)$

$$R(\omega) = 2\int_0^\infty x(t)\cos(\omega t)\,dt \qquad R(\omega) = 0$$

$$I(\omega) = 0 \qquad I(\omega) = -2\int_0^\infty x(t)\sin(\omega t)\,dt$$

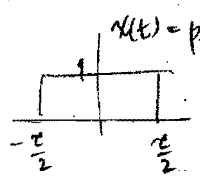
$\Downarrow$ $\qquad$ $\Downarrow$

If $x(t)$ is EVEN, $\qquad$ If $x(t)$ is ODD,

$$X(\omega) = 2\int_0^\infty x(t)\cos(\omega t)\,dt \qquad X(\omega) = -j2\int_0^\infty x(t)\sin(\omega t)\,dt$$

EXAMPLE

rectangular pulse . $x(t) = p_\tau(t) = \begin{cases} 1 & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0 & \text{o.w.} \end{cases}$



$\leftarrow$ Even fn. of t $\quad \therefore \quad X(\omega) = 2\int_0^{\tau/2} 1\cos(\omega t)\,dt$

$$= \frac{2}{\omega}\sin(\omega t)\Big|_{t=0}^{t=\tau/2}$$

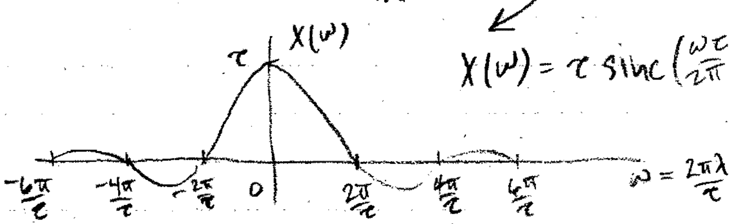
$$= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

recall $\mathrm{sinc}(\lambda) = \dfrac{\sin(\pi\lambda)}{\pi\lambda}$

$$= \tau \cdot \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} \quad \pi\lambda = \frac{\omega\tau}{2}$$

$X(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$

$$= \tau \cdot \mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$



10.01.03 Ch4 Fourier Series / Fourier Transform

BANDLIMITED SIGNALS.

$x(t)$ is BANDLIMITED if $X(\omega) = 0 \quad \omega > B$

positive integer (bandwidth)

Notes:

bandlimited signals can NOT be time limited

$\therefore$ finite $B \rightarrow$ infinite time span (duration)!

likewise,

finite time duration $\rightarrow$ infinite bandwidth (B)!

PROPERTIES OF FOURIER TRANSFORM.

LINEARITY

$$a x(t) + b v(t) \leftrightarrow a X(\omega) + b V(\omega)$$

(result of linearity of integration)

LEFT/RIGHT TIME SHIFT

$$x(t-c) \longleftrightarrow X(\omega) e^{-j\omega c}$$

TIME SCALING

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right) \qquad a > 0$$

Note: $0 < a < 1$ time expansion $\rightarrow$ frequency compression

$a > 1$ time compression $\rightarrow$ frequency expansion

time-limited vs. band-limited.

10.01.03 Ch4 Fourier Series / Fourier Transform.

TIME REVERSAL

$$x(-t) \leftrightarrow X(-\omega)$$

Note: if $x(t)$ is real valued, then $X(-\omega) = \overline{X(\omega)}$

$$\therefore \quad x(-t) \longleftrightarrow \overline{X(\omega)} \qquad x(t) \text{ is REAL.}$$

MULTIPLICATION BY POWER OF t

$$t^n x(t) \longleftrightarrow (j)^n \frac{d^n}{d\omega^n} X(\omega)$$

MULTIPLICATION BY A COMPLEX EXPONENTIAL

$$x(t)\, e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0) \qquad \text{frequency modulation}$$

$$\downarrow$$

$$x(t) \cos(\omega_0 t) \longleftrightarrow \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

$$x(t) \sin(\omega_0 t) \longleftrightarrow \frac{j}{2}\left[X(\omega+\omega_0) - X(\omega-\omega_0)\right]$$

TIME DOMAIN DIFFERENTIATION / INTEGRATION

$$\frac{d^n}{dt^n} x(t) \longleftrightarrow (j\omega)^n X(\omega)$$

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \longleftrightarrow \frac{1}{j\omega} X(\omega)$$

TIME DOMAIN CONVOLUTION / MULTIPLICATION

$$x(t) * v(t) \longleftrightarrow X(\omega) V(\omega)$$

$$x(t) v(t) \longleftrightarrow \frac{1}{2\pi}\left[X(\omega) * V(\omega)\right]$$

DUALITY

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

$$\uparrow$$

if we know FT in one direction, can apply it to similar signals

10·06·03 Ch 4 Fourier Series | Fourier Transform

PARSEVAL'S THEOREM

$$\int_{-\infty}^{\infty} x(t)\,v(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}\,V(\omega)\,d\omega$$

GENERALIZED FOURIER TRANSFORM

Some common fns don't have FT in ordinary sense $\left\{\begin{array}{l} u(t) \\ \cos(\omega_0 t) \\ \sin(\omega_0 t)\end{array}\right.$ $\Downarrow$ Generalized Fourier Transform.

$\uparrow$ $x(t) = 1$ $\forall t$ is not absolutely integrable

By conventional approach, $\delta(t) \longleftrightarrow 1 \quad -\infty < \omega < \infty$

Using DUALITY property $X(t) \longleftrightarrow 2\pi x(-\omega)$

if $x(t) = \delta(t) \longrightarrow X(\omega) = 1 \quad -\infty < \omega < \infty$

then by DUALITY $X(t) = 1 \quad -\infty < t < \infty \longrightarrow 2\pi x(-\omega) = 2\pi\delta(\omega)$

$\therefore \quad \underset{\text{constant in time}}{1} \longleftrightarrow \underset{\text{impulse in frequency}}{2\pi\delta(\omega)}$

other pairs.

$$\cos(\omega_0 t) \longleftrightarrow \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

$$\sin(\omega_0 t) \longleftrightarrow j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$$

$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

LTI system

$$x_c(t) = A e^{j(\omega_0 t + \theta)} \qquad -\infty < t < \infty$$

$\downarrow$

$$y_c(t) = x_c(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, x_c(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, A e^{j(\omega_0(t-\lambda)+\theta)}\, d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, e^{-j\omega_0\lambda}\, d\lambda \left[A e^{j(\omega_0 t+\theta)}\right]$$

$$= x_c(t) \int_{-\infty}^{\infty} h(\lambda)\, e^{-j\omega_0\lambda}\, d\lambda$$

$$= x_c(t)\; H(\omega_0)$$

complex fn. (magnitude & phase)

eigenfunction

10.06.03 Ch 5 Frequency Domain Analysis.

LTI Continuous. time system

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda)\, d\lambda = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda)\, d\lambda$$

Note: throughout Ch 5, assumed that $h(t)$ is absolutely integrable.

$\Downarrow$

$h(t) \longrightarrow H(\omega)$ exists.

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= |H(\omega)| e^{j \measuredangle H(\omega)}$$

$\therefore$ Assuming $H(\omega)$ exists, given

$$x(t) = A\cos(\omega_0 t + \theta) \quad -\infty < t < \infty$$

$\downarrow$

$\boxed{h(t) \leftrightarrow H(\omega)}$ LTI System - see opposite page.

$\downarrow$

$$y(t) = A|H(\omega_0)| \cos(\omega_0 t + \theta + \measuredangle H(\omega_0)) \quad -\infty < t < \infty$$

$H(\omega)$ frequency response $\left\{ \begin{array}{l} |H(\omega)| \text{ magnitude fn} \\ \measuredangle H(\omega) \text{ phase fn.} \end{array} \right.$

$\downarrow$

$H(\omega)|_{\omega=\omega_0} = H(\omega_0)$ ~ frequency response at particular frequency

RESPONCE TO PERIODIC INPUTS

periodic $\to c_k = f(k\omega_0) \to H(k\omega_0)$

Suppose $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad -\infty < t < \infty$

$\downarrow$

$\boxed{h(t) \leftrightarrow H(\omega)}$

$\downarrow$

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k e^{jk\omega_0 t} \quad -\infty < t < \infty$$

$\therefore$ if $x(t+T) = x(t)$
then $y(t+T) = y(t)$ $\Big\}$ response to periodic input w/ fund period T is periodic w/ fund. period T

10.06.03 Ch 5 Frequency Domain Analysis.

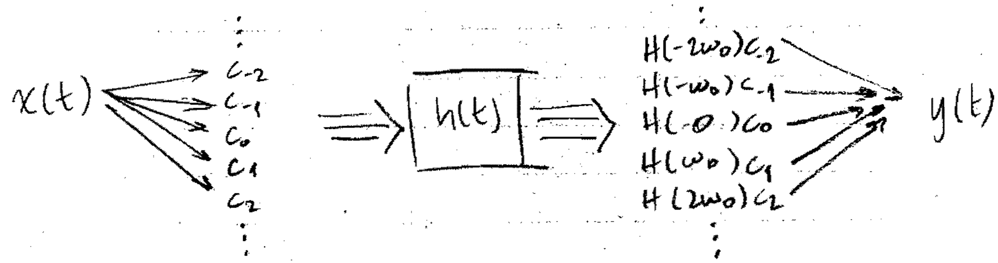
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0)\, c_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} |H(k\omega_0)|\,|c_k|\, e^{j(k\omega_0 t + \angle c_k + \angle H(k\omega_0))}$$

$\uparrow$ magnitude scaled (gain/attenuation).

$\uparrow$ phase shifted (linear delay or distortion) group delay.

∴ Response to PERIODIC INPUT is as if



Remember $x(t) * h(t) \longrightarrow X(\omega)H(\omega)$

$\uparrow$ (line spectra for PERIODIC signal)

RESPONSE TO APERIODIC INPUTS.

$$x(t) \rightarrow X(\omega)$$

$$\Downarrow$$

$$\boxed{h(t) \leftrightarrow H(\omega)}$$

$$\Downarrow$$

$$Y(\omega) = X(\omega)H(\omega)$$

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)|$$

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

$$\downarrow$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)\, e^{j\omega t}\, d\omega$$

$\uparrow$ note error in text p.216

$\uparrow$ assumes no initial energy in system prior to application of $x(t)$.

$$H(\omega) = p_{2B}(\omega)\, e^{-j\omega t_d} \quad -\infty < \omega < \infty$$

we know $\frac{\tau}{2\pi} \operatorname{sinc} \frac{\tau t}{2\pi} \leftrightarrow p_\tau(\omega)$

$$\Downarrow$$

$$p_{2B}(\omega) \rightarrow \frac{\cancel{2}B}{\cancel{2}\pi} \operatorname{sinc} \frac{\cancel{2}Bt}{\cancel{2}\pi} = \frac{B}{\pi} \operatorname{sinc}\left(\frac{B}{\pi} t\right)$$

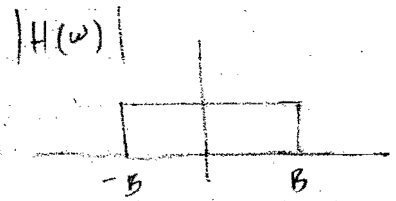
$$\downarrow$$

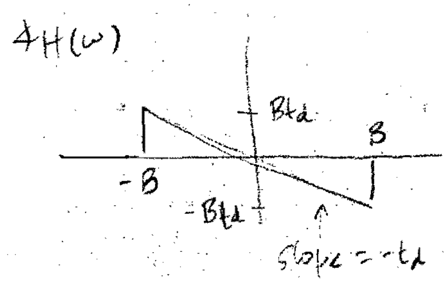
$$p_{2B}(\omega)\, e^{-j\omega t_d} \rightarrow \frac{B}{\pi} \operatorname{sinc}\left(\frac{B}{\pi}(t-t_d)\right)$$

$\therefore$

$$H(\omega) = p_{2B}(\omega)\, e^{-j\omega t_d} \longrightarrow h(t) = \frac{B}{\pi} \operatorname{sinc}\left(\frac{B}{\pi}(t-t_d)\right)$$

$$\downarrow$$



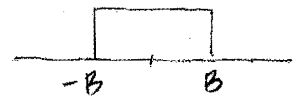


10-08-03 Ch 5 Frequency Domain Analysis

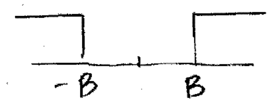
ANALYSIS OF IDEAL FILTERS.

MAGNITUDE FUNCTIONS:

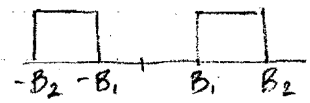
Ideal Lowpass . $|H(\omega)| = \begin{cases} 1 & -B \le \omega \le B \\ 0 & |\omega| > B \end{cases}$



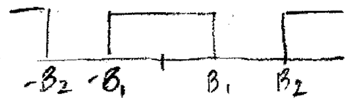
Ideal Highpass $|H(\omega)| = \begin{cases} 0 & -B < \omega < B \\ 1 & |\omega| \ge B \end{cases}$



Ideal Bandpass $|H(\omega)| = \begin{cases} 1 & B_1 \le \omega \le B_2 \\ 0 & \text{o.w.} \end{cases}$



Ideal Bandstop $|H(\omega)| = \begin{cases} 0 & B_1 \le \omega \le B_2 \\ 1 & \text{o.w.} \end{cases}$



PHASE FUNCTION

linear phase $\longrightarrow$ no phase distortion

$\Downarrow$

$\angle H(\omega) = -\omega t_d$ $\quad \forall \omega$ in passband

t_d - fixed positive delay

if ω_0 is in passband

$$x(t) = A\cos(\omega_0 t) \longrightarrow y(t) = A|H(\omega_0)|\cos(\omega_0(t - t_d))$$
$-\infty < t < \infty \qquad\qquad -\infty < t < \infty$

linear phase $\rightarrow$ time delay of t_d seconds.

IDEAL LINEAR PHASE LOWPASS FILTER

$\angle H(\omega) = -\omega t_d \qquad |\omega| \le B$

$|H(\omega)| = 1 \qquad |\omega| \le B$

$$H(\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| \le B \\ 0 & |\omega| > B \end{cases}$$

$= p_{2B}(\omega) e^{-j\omega t_d} \quad -\infty < \omega < \infty$

Given $H(\omega) = p_{2B}(\omega) e^{-j\omega t_d} \xrightarrow[\text{F.T.}]{} h(t) = \frac{B}{\pi}\,\text{sinc}\left[\frac{B}{\pi}(t - t_d)\right] \quad -\infty < t < \infty$

impulse response of ideal lowpass filter (Note: noncausal!)

$\leftarrow$ SEE OTHER SIDE FOR DETAILS $H(\omega) \rightarrow h(t)$

10.15.03 Ch 5 Frequency Domain Analysis.

skip

Note: In channel equalization, we attempt to restore linear phase to corrupted (received) signal.

$$x(t) \rightarrow \boxed{C(\omega)} \rightarrow v(t) \qquad v(t) \rightarrow \boxed{G(\omega)} \rightarrow y(t)$$

if $C(\omega)$ has nonlinear phase (distortion) [RF link?] → then $v(t)$ may be distorted version of $x(t)$

design $G(\omega)$ s.t.
$$C(\omega)G(\omega) = H(\omega)$$
linear phase

$$G(\omega) = \frac{H(\omega)}{C(\omega)}$$

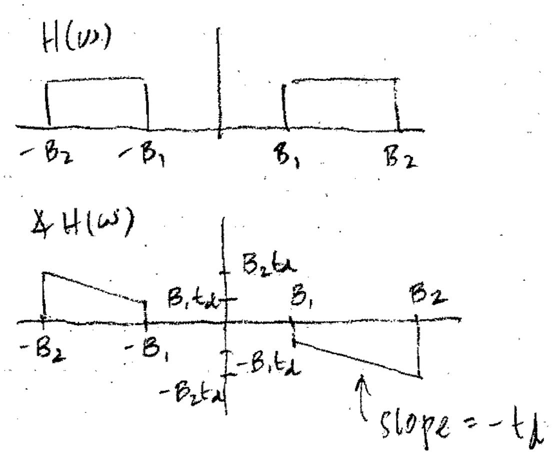
IDEAL LINEAR-PHASE BANDPASS FILTER

$$\measuredangle H(\omega) = -\omega t_d \qquad B_1 \le |\omega| \le B_2$$

$$|H(\omega)| = 1 \qquad B_1 \le |\omega| \le B_2$$

$$H(\omega) = \begin{cases} e^{-j\omega t_d} & B_1 \le |\omega| \le B_2 \\ 0 & \text{o.w.} \end{cases}$$

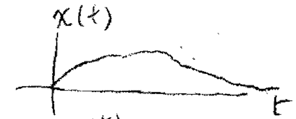
∴



* See supplemental notes on Response to Nonsinusoidal Input. Review during recitation

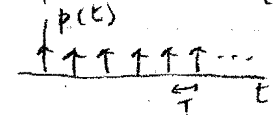
10.15.03 Ch 5 Frequency Domain Analysis

SAMPLING $x(t) \rightarrow x(nT) \rightarrow x[n]$

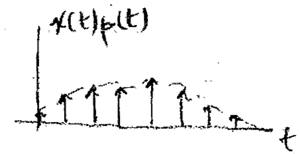


Impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

T sampling interval



$$x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$



$$x(t)p(t) \xrightarrow{FT} X(\omega) * P(\omega) \qquad P(\omega) = ?$$

Note: $p(t)$ is periodic $\rightarrow$ Fourier Series.

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$\omega_s = \frac{2\pi}{T}$ rad/sec sampling frequency $(f_s = \frac{1}{T} = \frac{\omega_s}{2\pi}$ Hz$)$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t}\,dt \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t}\,dt$$

$\delta(t)$ because integral is only over one period.

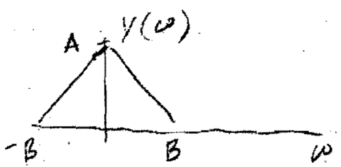
$$= \frac{1}{T}\left[e^{-jk\omega_s t}\right]\Big|_{t=0}$$

$$= \frac{1}{T}$$

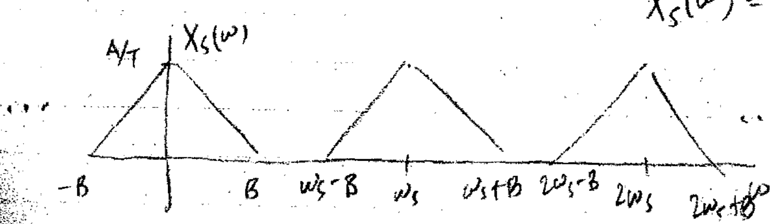
$$\Downarrow$$

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_s t} \longrightarrow x(t)p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} x(t) e^{jk\omega_s t}$$

spectral (frequency) modulation



$$X_s(\omega) = X(\omega) * P(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega - k\omega_s)$$



Note: $x_s(t) = x(t)p(t)$

if $\omega_s - B > B \rightarrow$ NO ALIASING $(\omega_s > 2B)$

if $\omega_s - B < B \rightarrow$ ALIASING $(\omega_s < 2B)$

10.15.03 Ch 5. Frequency Domain Analysis

If $|X(\omega)| = 0 \quad \omega > B$ strictly band-limited

then if $\omega_s \geq 2B$ $x(t)$ can be perfectly (exactly) reconstructed from $x_s(t)$ (by simple lowpass filtering)

$2B \sim$ Nyquist sampling freq.

INTERPOLATION FILTER

$$H(\omega) = \begin{cases} T & |\omega| \leq B \\ 0 & \text{o.w.} \end{cases}$$

$$h(t) = \frac{BT}{\pi} \operatorname{sinc}\left(\frac{B}{\pi} t\right) \quad -\infty < t < \infty$$

10.20.03 Ch 7 Discrete-Time Fourier Analysis

✓ Fourier Series (FS) - Periodic Waveforms

$$x(t) = x(t+T) \rightarrow x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad -\infty < t < \infty$$

$$\left(\omega_0 = \frac{2\pi}{T}\right) \qquad c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

✓ Fourier Transform (FT) - Aperiodic Waveforms.

$$x(t) \rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Discrete-Time Fourier Transform (DTFT) - Discrete-time Waveforms

$$x[n] \rightarrow X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Note: Discrete in time / Continuous in frequency (Ω) $\quad \underset{\text{DTFT}}{\Omega} \neq \underset{\text{FT}}{\omega}$

DTFT exists if $x[n]$ is absolutely summable, i.e., $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

Note: any time limited signal satisfies this condition $\rightarrow \sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-N}^{N} |x[n]| < \infty.$

10·20·03 Ch 7 Discrete Time Fourier Analysis

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ ($e^{-jn2\pi} = 1$ all integers n)

↑ periodic fn of Ω w/ period 2π. → $X(\Omega + 2\pi) = X(\Omega)$

∴ $X(\Omega)$ is completely determined over any 2π interval

$0 \le \Omega \le 2\pi$ / $-\pi \le \Omega \le \pi$

$X(\Omega)$ complex-valued in general.

$$\begin{aligned} X(\Omega) &= R(\Omega) + jI(\Omega) \\ &= \sum x[n] e^{-j\Omega n} \\ &= \sum x[n] (\cos(\Omega n) - j\sin(\Omega n)) \\ &= \sum_{n=-\infty}^{\infty} x[n] \cos(\Omega n) + j\left(-\sum_{n=-\infty}^{\infty} x[n] \sin(\Omega n)\right) \end{aligned}$$

$$\left\{\begin{aligned} R(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] \cos(\Omega n) \\ I(\Omega) &= -\sum_{n=-\infty}^{\infty} x[n] \sin(\Omega n) \end{aligned}\right.$$

$$|X(\Omega)| = (R^2(\Omega) + I^2(\Omega))^{1/2}$$

$$\angle X(\Omega) = \tan^{-1}\left(\frac{I(\Omega)}{R(\Omega)}\right)$$

$$X(\Omega) = |X(\Omega)| e^{j\angle X(\Omega)}$$

$|X(\Omega)|$ & $\angle X(\Omega)$ are periodic in Ω
∴ need be defined over 2π interval.

If $x[n]$ is real-valued → $|X(-\Omega)| = |X(\Omega)|$
$\angle X(-\Omega) = -\angle X(\Omega)$

If $x[n]$ is even $(x[-n] = x[n])$ → $X(\Omega) = x[0] + 2\sum_{n=1}^{\infty} x[n] \cos(\Omega n)$

If $x[n]$ is odd $(x[-n] = -x[n])$ → $X(\Omega) = x[0] - 2j\sum_{n=1}^{\infty} x[n] \sin(\Omega n)$

10.22.03 Ch 7 Discrete Time Fourier Analysis

$$X(\Omega + 2\pi) = X(\Omega)$$

throughout text → use interval $-\pi \leq \Omega \leq \pi$

remember $\Omega \neq \omega$ · highest frequency in DTFT is at $\pm\pi$ (i.e. $\Omega = \pm\pi$)

Must recognize - FT $-\infty < \omega < \infty$
DTFT $-\pi \leq \Omega \leq \pi$

sampling interval T is normalized to unity
(actual frequency range is function of samp. freq $= 1/T$)
$-\pi = -f_s/2$ $\pi = f_s/2$

INVERSE DTFT

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

PROPERTIES OF DTFT - see Tables 7.1 & 7.2 pp. 308-309

10.27.03

DISCRETE FOURIER TRANSFORM (DFT)

DISCRETE IN TIME
DISCRETE IN FREQUENCY

given $x[n] = 0$ $n < 0$ & $n \geq N$

N-pt DFT of $x[n]$ → $X_k = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$ $k = 0, 1, \ldots, N-1$

$\{X_0, X_1, \ldots, X_{N-1}\}$ completely represent $x[n]$

Complex Valued → $X_k = |X_k|e^{j\angle X_k}$
$= R_k + jI_k$

DFT always exists since $\sum_{n=0}^{N-1}|x[n]| < \infty$

10.27.03 Ch 7 Discrete Time Fourier Analysis

INVERSE DFT $\quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \quad n = 0, 1, \ldots, N-1$

RELATION TO DTFT

$$X_k = X(\Omega)\Big|_{\Omega = \frac{2\pi k}{N}} = X\left(\frac{2\pi k}{N}\right)$$

$$k = 0, 1, \ldots, N-1$$

Go over Examples ~~7.10~~ 4.11 and 7.11 in text.

skip

DFT OF TRUNCATED SIGNAL

$$\tilde{x}[n] = \begin{cases} x[n] & n = 0, 1, \ldots, N-1 \\ 0 & n \geq N \end{cases}$$

$$\text{if } p\left[n - \frac{N-1}{2}\right] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{o.w.} \end{cases}$$

$$\downarrow$$

$$\text{then } \tilde{x}[n] = x[n] \cdot p\left[n - \frac{N-1}{2}\right]$$

$$\Downarrow$$

$$\tilde{X}(\Omega) = X(\Omega) * P(\Omega)$$

$$\downarrow$$

$$\tilde{X}_k = \left[X(\Omega) * P(\Omega)\right]_{\Omega = \frac{2\pi k}{L}} \quad k = 0, 1, \ldots, L-1$$

$\uparrow$ L-pt DFT of $\tilde{x}[n]$

SYSTEM ANALYSIS VIA DTFT/DFT

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} h[i]\, x[n-i]$$

assuming $h[n]$ is absolutely summable $\left(\sum_{n=-\infty}^{\infty} |h[n]| < \infty\right)$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n}$$

$$\rightarrow Y(\Omega) = H(\Omega) X(\Omega)$$

$$|Y(\Omega)| = |H(\Omega)|\,|X(\Omega)| \qquad \angle Y(\Omega) = \angle H(\Omega) + \angle X(\Omega)$$

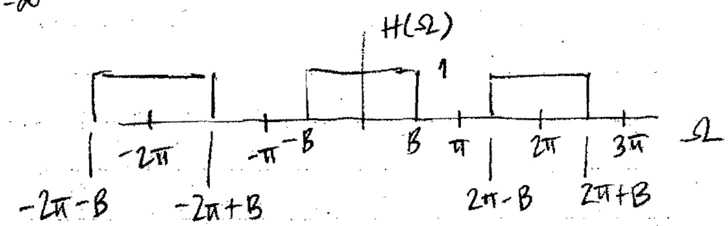
10.27.03 Ch 7 Discrete Time Fourier Analysis.

RESPONSE TO SINUSOIDAL INPUT

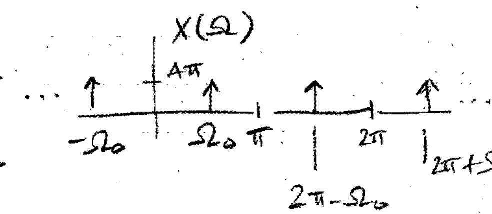
$$x[n] = A\cos(\Omega_0 n + \theta) \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$\Omega_0 \geq 0$$
$$\downarrow$$
$$y[n] = A|H(\Omega_0)|\cos(\Omega_0 n + \theta + \measuredangle H(\Omega_0))$$

Suppose $H(\Omega) = \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$ periodic in 2π

not a "true" LPF



given $x[n] = A\cos(\Omega_0 n)$ $n = 0, \pm 1, \pm 2, \ldots$ $\xrightarrow{\text{DTFT}}$



$$y[n] = x[n] * h[n]$$
$$\downarrow$$
$$Y(\Omega) = X(\Omega)H(\Omega)$$

if $\Omega_0 \leq B$ $\rightarrow Y(\Omega) = X(\Omega)H(\Omega) = X(\Omega) \longrightarrow y[n] = x[n] = A\cos(\Omega_0 n)$

if $B < \Omega_0 < \pi$ $\rightarrow Y(\Omega) = X(\Omega)H(\Omega) = 0 \longrightarrow y[n] = 0$

$\therefore$ $y[n] = A\cos(\Omega_0 n)$ if $2\pi k - B \leq \Omega_0 \leq 2\pi k + B$

P41 - unit-pulse response of $H(\Omega) = \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$

$$\downarrow$$
$$h[n] = \frac{B}{\pi}\,\mathrm{sinc}\left(\frac{B}{\pi}n\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

$h[n]$ ~ noncausal $\rightarrow$ no real-time implementation
$\rightarrow$ operate off-line on stored and current data.

10.27.03 Ch 7 Discrete-Time Fourier Analysis. skip.

DFT SYSTEM ANALYSIS

if $x[n] = 0 \quad n<0$ and $n \geq N$
and $h[n] = 0 \quad n<0$ and $n > Q$

then $y[n] = h[n] * x[n] = \sum_{i=0}^{\infty} h[i]\, x[n-i] \qquad n \geq 0$

$$= \sum_{i=0}^{Q} h[i]\, x[n-i] \qquad n \geq 0$$

$$y[n] = 0 \quad \text{all } n \geq N+Q$$

In DFT domain - zero pad $x[n]$ & $h[n]$ to length $N+Q$

$$\rightarrow \quad x[n] = 0 \qquad n = N, N+1, \ldots, N+Q-1$$
$$h[n] = 0 \qquad n = Q, Q+1, \ldots, N+Q-1$$

$\Downarrow$ $(N+Q)$-pt DFTs

$$X_k = \sum_{n=0}^{N+Q-1} x[n]\, e^{-j2\pi kn/(N+Q)} \qquad k = 0, 1, \ldots, N+Q-1$$
$$H_k = \sum_{n=0}^{N+Q-1} h[n]\, e^{-j2\pi kn/(N+Q)} \qquad k = 0, 1, \ldots, N+Q-1$$

$\downarrow$

$$Y_k = H_k X_k \qquad k = 0, 1, \ldots, N+Q-1$$

IDFT $\rightarrow$ $y[n] = \frac{1}{N+Q} \sum_{k=0}^{N+Q-1} H_k X_k\, e^{j2\pi kn/(N+Q)} \qquad n = 0, 1, \ldots, N+Q-1$

if $x[n] \neq 0 \quad n \geq N$ and/or $h[n] \neq 0 \quad n > Q$

$\hookrightarrow$ $y[n]$ is only approximate expression

10.29.03 Ch 8 - Laplace Transform.

FT $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

For existence, $x(t)$ must be absolutely integrable.

If $x(t) = u(t)$ → not absolutely integrable
no Fourier Transform (in ordinary sense)

$\therefore X(\omega) = \int_0^{\infty} e^{-j\omega t} dt$ does not exist!
$(e^{-j\omega\infty} = ?)$

If $x(t) = e^{-\sigma t} u(t)$ → absolutely integrable
if $\sigma > 0$
→ F.T. exists. $X(\omega) = \int_0^{\infty} e^{-\sigma t} e^{-j\omega t} dt$
$= \int_0^{\infty} e^{-(\sigma + j\omega)t} dt$

let $s = \sigma + j\omega$ → $X(s) = \int_0^{\infty} e^{-st} dt$

complex number

$= \frac{1}{s}$ Laplace Transform of unit-step fn.

complex-valued.

$X(s) = \frac{1}{s}$ defined only for $Re\{s\} = \sigma > 0$
(not defined for $\sigma = 0$ or $\sigma < 0$)

$Re\{s\} = \sigma > 0$ Region of Convergence

In general
$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

the set of all complex numbers s for which $X(s)$ exists is REGION of CONVERGENCE

depends on given fn $x(t)$

10.29.03 Ch 8 - Laplace Transform

if $x(t) = e^{-bt}u(t)$

$\downarrow$

then $X(s) = \int_0^\infty e^{-bt}e^{-st}\,dt$

$= \int_0^\infty e^{-(b+s)t}\,dt$

$= -\frac{1}{(s+b)}\left[e^{-(s+b)t}\right]_{t=0}^{t=\infty}$

what is limit $e^{-(s+b)t}$ as $t \to \infty$?

$e^{-(s+b)t} = e^{-(\sigma+j\omega+b)t} \to 0$ as $t \to \infty$

if $\sigma + b > 0$

$\Downarrow$

ROC: $Re\{s\} = \sigma > -b$

As such $\to$ $X(s) = \frac{1}{s+b} \longleftrightarrow x(t) = e^{-bt}u(t)$

Relationship b/w Fourier Transform + Laplace Transform

$X(\omega) = X(s)\big|_{s=j\omega}$ valid iff $\sigma = 0 \in$ ROC of $X(s)$

PROPERTIES OF LAPLACE TRANSFORM

LINEARITY $ax(t) + bv(t) \leftrightarrow aX(s) + bV(s)$

RIGHT TIME SHIFT $x(t-c)u(t-c) \leftrightarrow e^{-cs}X(s)$

TIME SCALING $x(at) \leftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right)$

MULTIPLICATION by POWER of t $t^N x(t) \longleftrightarrow (-1)^N \frac{d^N}{ds^N}X(s)$

MULT BY EXPONENTIAL $e^{at}x(t) \longleftrightarrow X(s-a)$

SIN $x(t)\sin\omega t \longleftrightarrow \frac{j}{2}[X(s+j\omega) - X(s-j\omega)]$

COS $x(t)\cos\omega t \longleftrightarrow \frac{1}{2}[X(s+j\omega) + X(s-j\omega)]$

10.29.03 Ch 8 Laplace Transform

DIFFERENTIATION IN TIME $\quad \dot{x}(t) \longleftrightarrow sX(s) - x(0)$

$$\frac{d^N}{dt^N} x(t) = x^{(N)}(t) \leftrightarrow s^N X(s) - s^{N-1}x(0) - s^{N-2}\dot{x}(0) - \cdots - s x^{(N-2)}(0) - x^{(N-1)}(0)$$

INTEGRATION
$$\int_0^t x(\lambda)\,d\lambda \longleftrightarrow \frac{1}{s} X(s)$$

CONVOLUTION $\quad x(t) * v(t) \leftrightarrow X(s)V(s)$

INITIAL VALUE THM - $\quad x(0) = \lim_{s\to\infty} sX(s)$
(valid for $t=0, 0^+$)
NOT for $t=0^-$

$$\dot{x}(0) = \lim_{s\to\infty} \left[s^2 X(s) - s x(0)\right]$$

$$x^{(N)}(0) = \lim_{s\to\infty} \left[s^{N+1}X(s) - s^N x(0) - s^{N-1}\dot{x}(0) - \cdots - s x^{(N-1)}(0)\right]$$

FINAL VALUE THM $\quad \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Inverse Laplace Transform
$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

c is any real number for which the path from $s=c-j\infty$ to $s=c+j\infty$ lies in the ROC of $X(s)$

In practice, we seldom actually evaluate this integral
↓ - require a course in COMPLEX VARIABLES

Instead - we'll use Partial Fraction Expansion & known Laplace Transform pairs (for rational $X(s)$ - explained now)

11·3·03 Ch 8 · Laplace Transforms

Suppose $x(t) \rightarrow X(s) = \frac{B(s)}{A(s)} = \frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$

$\{b_M, \ldots, b_0, a_N, \ldots, a_0\} \in \mathbb{R}$
$M, N \in$ positive integers.

if $b_M \neq 0 \rightarrow B(s) \sim$ degree M
if $a_N \neq 0 \rightarrow A(s) \sim$ degree N

(Note: assumed factors $(s-k)$ common to $B(s)$ & $A(s)$ are already factored out)

$X(s) = \frac{B(s)}{A(s)}$ "rational fn of s" of order N

roots of denominator $(A(s))$ are values of s for which

$A(s) = 0$ as such $A(s) = a_N(s-p_1)(s-p_2)\cdots(s-p_N)$

"zeros" of $A(s) \longrightarrow \{p_1, p_2, \ldots, p_N\} \in \mathbb{R}$ or $\mathbb{C}$

poles of $X(s)$ = zeros of $A(s)$

Note: complex zeros (poles) occur in complex conjugate pairs.

Use MATLAB to determine roots of arbitrary $A(s)$
- complex conjugate pairs.

$$X(s) = \frac{B(s)}{a_N(s-p_1)(s-p_2)\cdots(s-p_N)}$$

poles of $X(s)$ = zeros of $A(s)$
make $X(s) \rightarrow \infty$ — make $A(s) \rightarrow 0$

if $M < N \rightarrow X(s)$ is strictly proper in s

DISTINCT POLES (NONREPEATING) $\rightarrow p_i \neq p_j,\ i \neq j$

PFE → partial fraction expansion.

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_N}{s-p_N} \quad \text{w/ } c_i = \left[(s-p_i)X(s)\right]_{s=p_i}$$

$i = 1, 2, \ldots, N$

$\{c_i\}\ i = 1, 2, \ldots, N \sim$ RESIDUES of $X(s)$

11.3.03 Ch 8 Laplace Transform.

c_i is real if p_i is real   complex c_i occur in complex conjugate pairs. (result of p_i & $\bar{p_i}$)

Note: to compute $\{c_i\} = \{[(s-p_i)X(s)]_{s=p_i}\}$
do not need to factor B(s)
(you do need to factor A(s) to resolve poles)

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_N}{s-p_N}$$

$\downarrow \mathcal{L}^{-1}\{\cdot\}$

$$x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_N e^{p_N t} \qquad t \geq 0$$

poles determine convergence properties of $x(t)$
each term represents a "mode" of convergence
- if $\mathrm{Re}(p_i) > 0 \rightarrow$ that mode grows w/o bound
- stability (studied later) requires $\mathrm{Re}(p_i) < 0 \ \forall i$
- $\mathrm{Re}(p_i) = 0 \rightarrow$ marginally stable

poles $\in$ LHP of s-plane

poles determine characteristics of the time variation of $x(t)$

if $c_i(p_i)$ are real $\rightarrow c_i e^{p_i t}$ is real

if $c_i(p_i)$ are complex $\rightarrow c_i e^{p_i t}$ & $\bar{c_i} e^{\bar{p_i} t}$
can be combined to produce real form

DISTINCT POLES w/ TWO OR MORE COMPLEX POLES.
(still distinct)

suppose $p_1 = \sigma + j\omega$ $(\omega \neq 0) \rightarrow \bar{p}_1 = \sigma - j\omega$ is also a pole. $\{p_1, \bar{p}_1\} \Downarrow \{c_1, \bar{c}_1\}$

$$X(s) = \frac{c_1}{s-p_1} + \frac{\bar{c}_1}{s-\bar{p}_1} + \frac{c_3}{s-p_3} + \cdots + \frac{c_N}{s-p_N}$$

$\downarrow \mathcal{L}^{-1}\{\cdot\}$

$$x(t) = c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} + c_3 e^{p_3 t} + \cdots + c_N e^{p_N t} \qquad t \geq 0$$

$$= 2|c_1| e^{\sigma t} \cos(\omega t + \angle c_1) + c_3 e^{p_3 t} + \cdots + c_N e^{p_N t}$$

$c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} = 2|c_1| e^{\sigma t} \cos(\omega t + \angle c_1)$

11·3·03 Ch 8 Laplace Transform

6.17 6.18

Examples ~~8.16~~ & ~~8.17~~ are pretty good demonstrations.

Alternative means of resolving complex poles –

suppose $X(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

↓ "completing the square"

$$X(s) = \frac{b_1 s + b_0}{(s+\frac{a_1}{2})^2 + a_0 - \frac{a_1^2}{4}}$$

(remember: $M<N$)

poles of $X(s)$ are complex (a la quadratic formula) iff $a_0 - \frac{a_1^2}{4} > 0$.

If $a_0 - \frac{a_1^2}{4} > 0 \rightarrow p_1, \bar{p}_1 = -\frac{a_1}{2} \pm j\omega$ w/ $\omega = \left(a_0 - \frac{a_1^2}{4}\right)^{1/2}$

$$\Downarrow$$

$$X(s) = \frac{b_1 s + b_0}{(s+\frac{a_1}{2})^2+\omega^2} = \frac{b_1(s+\frac{a_1}{2})}{(s+\frac{a_1}{2})^2+\omega^2} + \frac{(b_0 - \frac{b_1 a_1}{2})\cdot\frac{1}{\omega}\cdot\omega}{(s+\frac{a_1}{2})^2+\omega^2}$$

from look-up table ↓ ↓

$$x(t) = b_1 e^{-\frac{a_1 t}{2}}\cos(\omega t)\,u(t) + (b_0 - \tfrac{b_1 a_1}{2})\cdot\underline{\tfrac{1}{\omega}}\, e^{-\frac{a_1 t}{2}}\sin(\omega t)\,u(t)$$

↑ keep eye on this

In general, suppose

$$X(s) = \frac{B(s)}{[(s-\sigma)^2+\omega^2](s-p_3)(s-p_4)\cdots(s-p_N)}$$

$$= \frac{b_1 s + b_0}{(s-\sigma)^2+\omega^2} + \frac{c_3}{s-p_3} + \frac{c_4}{s-p_4} + \cdots + \frac{c_N}{s-p_N} \qquad \{b_0, b_1, c_3, c_4, \ldots, c_N\} \in \mathbb{R}$$

b_1 & b_0 could be determined by putting RHS ↑ above over common denominator

$$[(s-\sigma)^2+\omega^2](s-p_3)(s-p_4)\cdots(s-p_N)$$

and equating terms (Example ~~8.19~~ 6.20 in text)

11.10.03 Ch 8 Laplace Transform

REPEATED POLES $(M < N)$

$$X(s) = \frac{B(s)}{A(s)} = \frac{B(s)}{\underbrace{(s-p_1)^r}_{\text{repeated } r \text{ times}}(s-p_{r+1})\cdots(s-p_N)}$$

$$\downarrow$$

$$X(s) = \frac{c_1}{(s-p_1)} + \frac{c_2}{(s-p_1)^2} + \cdots + \frac{c_r}{(s-p_1)^r} + \frac{c_{r+1}}{(s-p_{r+1})} + \cdots + \frac{c_N}{(s-p_N)}$$

$$\{c_1, \ldots, c_r\} \rightarrow c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{ds^i}\left[(s-p_1)^r X(s)\right]\right]_{s=p_1}$$

$$i = \cancel{0}, 1, 2, \ldots, r-1$$

$$\therefore \quad c_r = \left[(s-p_1)^r X(s)\right]_{s=p_1}$$

$$c_{r-1} = \left[\frac{d}{ds}\left[(s-p_1)^r X(s)\right]\right]_{s=p_1}$$

$$\vdots$$

Note. $t^N e^{-bt} u(t) \leftrightarrow \frac{N!}{(s+b)^{N+1}}$ $\quad N = 1, 2, 3 \ldots$

$\{c_1, \ldots, c_r\}$ could also be determined by
- evaluating coefficients (w/ common denom & linear alg.)
- quadratic terms

If $M \geq N$

$$\rightarrow X(s) = \frac{B(s)}{A(s)} = \underbrace{Q(s)}_{\text{degree } M-N} + \underbrace{\frac{R(s)}{A(s)}}_{\text{expand using PFE}}$$

TRANSFORM OF INPUT/OUTPUT DIFF EQS

1st ORDER CASE $\quad \frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$

$$\downarrow \mathcal{L}\{\} \quad \downarrow \mathcal{L}\{\} \quad \downarrow \mathcal{L}\{\cdot\}$$

$$sY(s) - y(0^-) + aY(s) = bX(s)$$

$$\therefore \quad (s+a)Y(s) = y(0^-) + bX(s)$$

$$Y(s) = \frac{y(0^-)}{(s+a)} + \frac{b}{(s+a)} X(s)$$

11·10·03 Ch 8 · Laplace Transform

$$Y(s) = \underbrace{\frac{y(0^-)}{(s+a)}}_{\substack{\text{output due to}\\\text{initial condition}}} + \underbrace{\frac{b}{(s+a)}X(s)}_{\substack{\text{output due to}\\\text{forcing fn. }(x(t))}}$$

$[x(t)=\delta(t) \to X(s)=1]$

if no initial energy → $y(0^-)=0$ → $Y(s) = \frac{b}{(s+a)}X(s)$

$$= H(s)X(s)$$

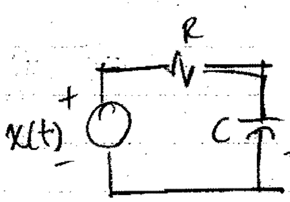
$H(s) = \frac{b}{s+a}$ transfer fn.

$\downarrow$

$h(t) = be^{-at}u(t)$

In general,

$H(s) = \frac{Y(s)}{X(s)}$ transfer fn. representation



$v_c(t) = y(t) \Rightarrow \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$

$$(s+\tfrac{1}{RC})Y(s) = y(0^-) + \tfrac{1}{RC}X(s)$$

$$\to Y(s) = \frac{y(0^-)}{(s+\frac{1}{RC})} + \frac{(1/RC)X(s)}{(s+\frac{1}{RC})}$$

if $x(t)=u(t) \to X(s)=\frac{1}{s} \Rightarrow$ final $y(t) = y(0^-)e^{-(1/RC)t} + 1 - e^{-(1/RC)t}$ $t \geq 0$

SECOND-ORDER CASE

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_1\frac{dx(t)}{dt} + b_0x(t)$$

$$s^2Y(s) - sy(0^-) - \dot{y}(0^-) + a_1sY(s) - a_1y(0^-) + a_0Y(s) = b_1sX(s) - b_1x(0^-) + b_0X(s)$$

$$Y(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1y(0^-)}{s^2+a_1s+a_0} + \frac{b_1s+b_0}{s^2+a_1s+a_0}X(s)$$

iff $y(0^-)=\dot{y}(0^-)=0$

then

$H(s) = \frac{b_1s+b_0}{s^2+a_1s+a_0}$

$Y(s) = H(s)X(s)$

11·10·03 ch 8 Laplace Transform.

N^{th} ORDER CASE – $\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{M} b_j \frac{d^j}{dt^j} x(t)$ $M \leq N$

assuming

$y(0^-) = \dot{y}(0^-) = \cdots = y^{(N)}(0^-) = 0 \Rightarrow Y(s) = \underbrace{\frac{b_M s^M + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}}_{H(s)} X(s)$

TRANSFER FN. REPRESENTATION

$$y(t) = h(t) * x(t) = \int_0^\infty h(\lambda) x(t-\lambda) d\lambda$$

$$Y(s) = H(s) X(s) \qquad (h(t) \leftrightarrow H(s))$$

$$\Downarrow$$

$$H(s) = \frac{Y(s)}{X(s)}$$

valid iff system is LTI

if not LTI $H(s)$ would have no meaning

- vary in time
- nonlinear effects

Any system given by input/output diff eq.

→ yields transfer fn ($H(s)$) ~ rational fn of s.

$$H(s) = \frac{b_M s^M + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$

(leading denominator coefficient is 1.)

$$H(s) = \frac{b_M (s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

$\{z_1, z_2, \ldots, z_M\}$ – ZEROS.

$\{p_1, p_2, \ldots, p_N\}$ – POLES.

RLC CIRCUITS

— $v(t) = Ri(t) \rightarrow V(s) = RI(s)$

— $\frac{dv(t)}{dt} = \frac{1}{C} i(t) \longrightarrow sV(s) - v(0) = \frac{1}{C} I(s)$

$V(s) = \frac{1}{Cs} I(s) + \frac{1}{s} v(0)$

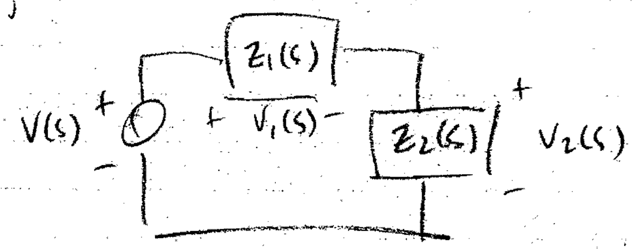
— $v(t) = L \frac{di(t)}{dt} \longrightarrow V(s) = sLI(s) - Li(0).$

skip

11·10·03 Ch 8 Laplace Transform

Resistor Impedance → R
Capacitor Impedance → 1/Cs
Inductor Impedance → Ls.

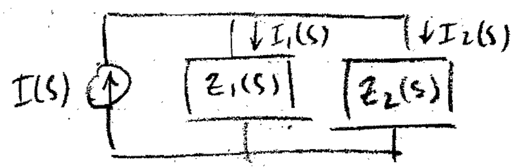
Voltage Divider -



$$V_1(s) = \frac{Z_1(s)}{Z_1(s)+Z_2(s)} V(s)$$

$$V_2(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} V(s)$$

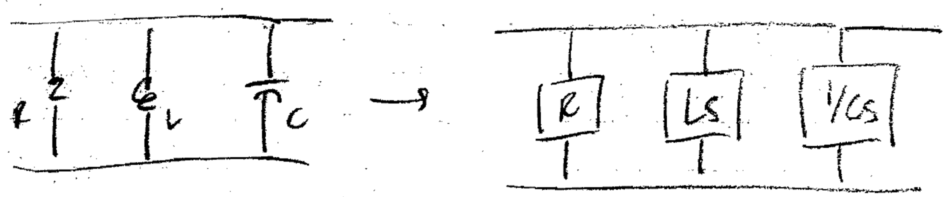
Current Divider



$$I_1(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} I(s)$$

$$I_2(s) = \frac{Z_1(s)}{Z_1(s)+Z_2(s)} I(s)$$

R‖L‖C ckt



Integrators → $y(t) = y(0) + \int_0^t x(t)\,dt$ $\quad x(t) \to \boxed{\int dt} \to y(t)$

$$Y(s) = \frac{1}{s} y(0) + \frac{1}{s} X(s)$$

if $y(0) = 0 \rightarrow Y(s) = \frac{1}{s} X(s)$ $\quad X(s) \to \boxed{\frac{1}{s}} \to Y(s)$

↑ INTEGRATE
↓ DERIVATIVE

Differentiator → $y(t) = \frac{dx(t)}{dt}$ $\quad x(t) \to \boxed{\frac{d}{dt}} \to y(t)$

$$Y(s) = sX(s) - x(0)$$

if $x(0) = 0 \rightarrow Y(s) = sX(s)$ $\quad X(s) \to \boxed{s} \to Y(s)$

skip

11·12·03 Ch 9 Transfer fn – System Anal.
(9.1, 9.4, 9.5)

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

stability requires → $Re\{p_i\} < 0 \quad i = 1, 2, \ldots, N$

∴ $h(t)$ converges to 0 as $t \to \infty$ iff $\forall p_i \in$ OLHP

(open left-half plane)

stable if $h(t) \to 0$ as $t \to \infty$

marginally stable if $|h(t)| \leq c$ for all t

→ $Re\{p_i\} \leq 0 \quad i = 1, 2, \ldots, N$

∴ can be poles (nonrepeated) on $j\omega$ axis.

unstable $|h(t)| \to \infty$ as $t \to \infty$

∴ at least 1 pole is in ORHP
or repeated poles on $j\omega$-axis.

Other models of stability -

* $h(t) \to 0$ as $t \to \infty$ iff $\int_0^\infty |h(t)|\,dt < \infty$
absolute integrability

* BIBO stable (bounded input → bounded output)
if $|x(t)| \leq c_1 \longrightarrow |y(t)| \leq c_2 \quad \forall t$

BIBO stable ≡ absolute integrability.

skip.

11.12.03 Ch 9 - Transfer fn - System Analysis.

$$H(s) = \frac{B(s)}{A(s)}$$

$$x(t) = C\cos\omega_0 t \qquad t \geq 0$$

$$\downarrow$$

$$X(s) = \frac{Cs}{s^2+\omega_0^2} = \frac{Cs}{(s+j\omega_0)(s-j\omega_0)}$$

↗ zero @ $s=0$

→ poles @ $s = \pm j\omega_0$

assuming no initial energy → $Y(s) = H(s)X(s)$

$$= \frac{B(s)}{A(s)} \cdot \frac{Cs}{(s+j\omega_0)(s-j\omega_0)}$$

$r(s)$ is polynomial in s

$$= \frac{r(s)}{A(s)} + \frac{c}{s-j\omega_0} + \frac{\bar{c}}{s+j\omega_0}$$

$$c = \left[(s-j\omega_0)Y(s)\right]\Big|_{s=j\omega_0}$$

$$= \left[\frac{Cs\,B(s)}{A(s)(s+j\omega_0)}\right]_{s=j\omega_0}$$

$$= \frac{jC\omega_0 B(j\omega_0)}{A(j\omega_0)(j2\omega_0)}$$

$$= \frac{C}{2}H(j\omega_0) \longrightarrow Y(s) = \frac{r(s)}{A(s)} + \frac{\left(\frac{C}{2}\right)H(j\omega_0)}{(s-j\omega_0)} + \frac{\left(\frac{C}{2}\right)\overline{H(j\omega_0)}}{(s+j\omega_0)}$$

$$y(t) = y_1(t) + \frac{C}{2}H(j\omega_0)e^{j\omega_0 t} + \frac{C}{2}\overline{H(j\omega_0)}e^{-j\omega_0 t}$$

$$= y_1(t) + C|H(j\omega_0)|\cos(\omega_0 t + \measuredangle H(j\omega_0)) \qquad t \geq 0$$

if system is stable → $y_1(t) \to 0$ as $t \to \infty$ } $y_1(t)$ is <u>transient</u> response

steady-state: $y_{ss}(t) = C|H(j\omega_0)|\cos(\omega_0 t + \measuredangle H(j\omega_0)) \quad t \geq 0$

$H(j\omega) = H(\omega)$ ⟶ $y_{ss}(t) = C|H(\omega_0)|\cos(\omega_0 t + \measuredangle H(\omega_0)) \quad t \geq 0$

∴ frequency response of stable system can be determined from $H(s)$

11·17·03 Ch 9 Transfer fn - System Anal.

Arbitrary inputs

$$X(s) = \frac{C(s)}{D(s)} \qquad H(s) = \frac{B(s)}{A(s)}$$

$$Y(s) = H(s)X(s) = \frac{B(s)C(s)}{A(s)D(s)}$$

$$= \frac{E(s)}{A(s)} + \frac{F(s)}{D(s)}$$

$$y(t) = y_1(t) + y_2(t)$$

form depends on poles of $\underline{\underline{H(s)}}$

$\Downarrow$

(if $h(t)$ is stable $y_1(t) \to 0$ as $t \to \infty$)

TRANSIENT RESPONSE

form depends on pole of $\underline{\underline{X(s)}}$

$\Downarrow$

(if $X(s)$ has poles on $j\omega$-axis $y_2(t) \not\to 0$ as $t \to \infty$)

STEADY-STATE RESPONSE.

Frequency Response Function

rational transfer fn. $H(s) = \frac{B(s)}{A(s)}$

given $x(t) = C\cos(\omega_0 t) \quad t \geq 0.$

$$y_{ss}(t) = C|H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0)) \quad t \geq 0$$

$$|H(\omega)|_{dB} = 20\log_{10}|H(\omega)| \quad (= 10\log_{10}|H(\omega)|^2)$$

↑ power spectrum

drill

Note:

$|H(\omega)|_{dB} < 0$ dB if $|H(\omega)| < 1$ attenuation

$|H(\omega)|_{dB} = 0$ dB if $|H(\omega)| = 1$

$|H(\omega)|_{dB} > 0$ dB if $|H(\omega)| > 1$ amplification

plots of $|H(\omega)|_{dB}$ vs. ω and $\angle H(\omega)$ vs. ω $\Big\}$ → Bode plots. (Bode diagrams.)

11.17.03 Ch 9 Transfer Fn - System Analysis.

First-order case (Single-pole - no zeros)

$$H(s) = \frac{k}{s+B} \rightarrow H(\omega) = \frac{k}{j\omega+B} \quad (= H(s)|_{s=j\omega})$$

$$|H(\omega)| = \frac{k}{\sqrt{\omega^2+B^2}} \quad (|H(\omega)|^2 = H(\omega)\overline{H(\omega)})$$

$$\angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{B}\right)$$

	$\omega = 0$	$\omega = B$	$\omega \to \infty$
$H(\omega)$	$H(0) = k/B$	$H(B) = k/2B = \frac{1}{2}H(0)$	$H(\infty) \to 0$
$\angle H(\omega)$	$\angle H(0) = 0°$	$\angle H(B) = -45°$	$\angle H(\infty) \to -90°$

3dB point (since $\frac{1}{2} \to 3dB$)

$|H(B)|_{dB} = |H(0)|_{dB} - 3dB$.

Characteristics
→ LPF
$\omega < B$ → passband $\therefore$ 3dB bandwidth = B rad/sec.
$\omega > B$ → stopband

→ No zero | single pole → poor LPF

— use MATLAB Bode script p. 467

Single-pole | single-zero

$$H(s) = \frac{s+C}{s+B} \rightarrow H(\omega) = \frac{j\omega+C}{j\omega+B}$$

$$|H(\omega)| = \sqrt{\frac{\omega^2+C^2}{\omega^2+B^2}}$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{C}\right) - \tan^{-1}\left(\frac{\omega}{B}\right)$$

$|H(0)| = \frac{C}{B} \longrightarrow |H(\infty)| \to 1$ w/ $C<B$ → HPF.

$\angle H(0) = 0° \longrightarrow \angle H(\infty) \to 0°$

max $\angle H(\omega)$ occurs b/t $\omega = C$ and $\omega = B$.

skip

11·23·03 Ch 9 Transfer Fn. - System Analysis.

SECOND ORDER SYSTEM.

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$k > 0,\ \zeta > 0,\ \omega_n > 0.$
(SYSTEM IS STABLE)

$$= \frac{k}{(s-p_1)(s-p_2)}$$

w/ $p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$
$p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$$\downarrow$$

$$|H(\omega)| = \frac{k}{|j\omega - p_1|\,|j\omega - p_2|}$$

$$\measuredangle H(\omega) = -\angle j\omega - p_1 - \angle j\omega - p_2$$

$\left\{ \begin{array}{l} p_1, p_2 \text{ are real} \\ \text{iff } \underline{\underline{\zeta \geq 1}} \end{array} \right.$

$|H(\omega)|$: $\omega = 0$ $|H(0)| = \left|\frac{k}{p_1 p_2}\right| = \frac{k}{\omega_n^2} \longrightarrow |H(\infty)| \to 0$

$\measuredangle H(\omega)$: $\measuredangle H(0) = 0° \longrightarrow \measuredangle H(\infty) \to -180°$

if $k = \omega_n^2 \to |H(0)| = 1 \longrightarrow$ LPF

w/ $\zeta = 1 \to$ 3dB bandwidth $= \sqrt{\sqrt{2}-1}\ \omega_n^2$

IF $0 < \zeta < 1 \to p_1$ & p_2 are complex w/ $p_1, p_2 = -\zeta\omega_n \pm j\omega_d$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\Downarrow$$

$$H(\omega) = \frac{k}{(j\omega + \zeta\omega_n + j\omega_d)(j\omega + \zeta\omega_n - j\omega_d)}$$

when $\zeta < \frac{1}{\sqrt{2}}$ $|H(\omega)|$ increases as ω goes from $0 \to \infty$ $|H(\infty)| \to 0$

when $\zeta \geq \frac{1}{\sqrt{2}}$ $|H(\omega)|$ decreases as ω goes from $0 \to \infty$

when $\zeta = \frac{1}{\sqrt{2}} \to$ RESONANCE. $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ resonant frequency

ch 8

11.23.03 Ch9 - Transfer Fn - System Analysis

Bode Plot Construction.

$$H(s) = \frac{A(s+C_1)(s+C_2)\cdots(s+C_M)}{s(s+B_1)(s+B_2)\cdots(s+B_{N-1})}$$ zeros @ $-C_1, -C_2, \ldots, -C_M$

poles @ $-B_1, -B_2, \ldots, -B_{N-1}$

$\downarrow$

$$H(\omega) = \frac{A(j\omega+C_1)(j\omega+C_2)\cdots(j\omega+C_M)}{j\omega(j\omega+B_1)(j\omega+B_2)\cdots(j\omega+B_{N-1})}$$

$$= \frac{K(j\frac{\omega}{C_1}+1)(j\frac{\omega}{C_2}+1)\cdots(j\frac{\omega}{C_M}+1)}{j\omega(j\frac{\omega}{B_1}+1)(j\frac{\omega}{B_2}+1)\cdots(j\frac{\omega}{B_{N-1}}+1)} \qquad K = \frac{AC_1C_2\cdots C_M}{B_1B_2\cdots B_{N-1}}$$

$$\therefore \; |H(\omega)|_{dB} = 20\log_{10}|K| + 20\log_{10}\left|j\frac{\omega}{C_1}+1\right| + \cdots + 20\log_{10}\left|j\frac{\omega}{C_M}+1\right|$$

$$-20\log_{10}|j\omega| - 20\log_{10}\left|j\frac{\omega}{B_1}+1\right| + \cdots + 20\log_{10}\left|j\frac{\omega}{B_{N-1}}+1\right|$$

$$\angle H(\omega) = \angle K + \angle\left(j\frac{\omega}{C_1}+1\right) + \cdots + \angle\left(j\frac{\omega}{C_M}+1\right)$$

$$-\angle j\omega - \angle\left(j\frac{\omega}{B_1}+1\right) - \angle\left(j\frac{\omega}{B_{N-1}}+1\right)$$

clip

Notes: $\underline{K}$: $|K|_{dB} = 20\log_{10}|K|$

$\angle K = 0$ or $180°$

$K>0$ or $K<0$

$\underline{(j\omega T+1)}$: $|j\omega T+1|_{dB} = 20\log_{10}\sqrt{\omega^2T^2+1}$

if $\omega_{cf} = \frac{1}{T} \rightarrow \omega_{cf}T = 1$

for $\omega < \omega_{cf} \rightarrow |j\omega T+1|_{dB} = 0\,dB$

for $\omega > \omega_{cf} \rightarrow |j\omega T+1|_{dB} = 20\log_{10}(\omega T)$ $+20\,dB/decade$

Figure 9.25 $\angle(j\omega T+1) = 0°$ very small frequencies

$\angle(j\omega T+1) = 90°$ very large frequencies

b/w $0.1\omega_{cf}$ & $10\omega_{cf}$ $\angle(j\omega T+1)$ increases by $45°/decade$

11.23.03 Ch 9 Transfer Fn - System Analysis

$(j\omega T+1) \to$ zeros - $|j\omega T+1|_{dB} > 0$ $\angle(j\omega T+1) > 0$

poles - $|j\omega T+1|_{dB} < 0$ $\angle(j\omega T+1) < 0$

$(j\omega)$: $\begin{cases} |j\omega|_{dB} = 20\log_{10}(\omega) & \text{no breakpoints} \\ \angle(j\omega) = 90° & \text{no shift.} \end{cases}$

as a zero

as a pole $\to \begin{cases} |j\omega|_{dB} = -20\log_{10}(\omega) \\ \angle(j\omega) = -90° \end{cases}$

Complex poles/zeros -> see text

skip

Ch 11 - z-transform / DTS

Discrete Time Systems

discrete-time counterpart to Laplace Transform

Given $x[n]$

$\to$ DTFT $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

to ensure convergence (absolute summability), add p^{-n} factor as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] p^{-n} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] (pe^{j\Omega})^{-n}$$

let $z = pe^{j\Omega}$

$\to X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ two-sided

One sided z transform

$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$ well-defined if $z \in$ ROC.

12-01-03 Ch 11 - Z-Transform / DTS

$X(\Omega) = X(z)|_{z=e^{j\Omega}}$ iff ROC of $X(z)$ includes all complex numbers z s.t. $|z| = 1$

↑ unit circle in complex plane.

PROPERTIES

Basic relationships

$$\begin{cases} x[n] = \delta[n] \to X(z) = 1 \\ x[n] = \delta[n-q] \to X(z) = z^{-q} \\ x[n] = u[n] \to X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \\ x[n] = a^n u[n] \to X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}} \end{cases}$$

LINEARITY

$$ax[n] + bv[n] \to aX(z) + bV(z)$$

RIGHT SHIFT of $x[n]u[n]$

$$x[n-q]u[n-q] \to z^{-q}X(z)$$

RIGHT SHIFT of $x[n]$

$$x[n-q] \to z^{-q}X(z) + x[-q] + z^{-1}x[-q+1] + \cdots + z^{-q+1}x[-1]$$

$\therefore$ if $x[n] = 0$, $n = -1, -2, \ldots, -q$

then $x[n-q] \to z^{-q}X(z)$

LEFT SHIFT of $x[n]$

$$x[n+q] \to z^{q}X(z) - x[0]z^{q} - x[1]z^{q-1} + \cdots + x[q-1]z$$

MULTIPLICATION BY n & n^2

$$nx[n] \to -z\frac{d}{dz}X(z)$$

$$n^2x[n] \to z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z)$$

12.01.03 Ch 11 Z-Transform / DTS

MULTIPLICATION BY a^N

$$a^n x[n] \rightarrow X\left(\frac{z}{a}\right)$$

MULTIPLICATION BY $\cos \Omega n$ & $\sin \Omega n$

$$x[n] \cos \Omega n \rightarrow \frac{1}{2}\left[X(e^{j\Omega} z) + X(e^{-j\Omega} z)\right]$$

$$x[n] \sin \Omega n \rightarrow \frac{j}{2}\left[X(e^{j\Omega} z) - X(e^{-j\Omega} z)\right]$$

SUMMATION

$$v[n] = \sum_{i=0}^{n} x[i] \rightarrow V(z) = \frac{z}{z-1} X(z)$$

CONVOLUTION

$$x[n] * v[n] \rightarrow X(z)V(z)$$

INITIAL VALUE THM.

$$x[q] = \lim_{z\to\infty} \left[z^q X(z) - z^q x[0] - z^{q-1} x[1] \cdots - z\, x[q-1]\right]$$

FINAL VALUE THEOREM

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1) X(z)$$

Inverse Z-Transform

$$x[n] = \frac{1}{2\pi j} \int X(z)\, z^{n-1}\, dz \qquad (\cancel{x}?)$$ error in text

Not generally going to use this form → use PFE.

12-01-03 Ch 11 Z-transform / DTS.

Since $X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2} + \ldots$

$\downarrow$

One way to obtain $x[n]$ is by long division...

$$X(z) = \frac{B(z)}{A(z)}$$

$A(z), B(z)$ written in descending powers of z as in

$$X(z) = \frac{z^2 - 1}{z^3 + 2z + 4}$$

$$\begin{array}{r|l}
 & z^{-1} + 0z^{-2} - 3z^{-3} - 4z^{-4} \\
z^3 + 0z^2 + 2z + 4 & z^2 + 0z - 1 \\
 & z^2 + 0z + 2 + 4z^{-1} \\
 & -3 - 4z^{-1} \\
 & -3 + 0z^{-1} - 6z^{-2} - 12z^{-3} \\
 & -4z^{-1} + 6z^{-2} + 12z^{-3} \\
 & -4z^{-1} + 0z^{-2} - 8z^{-3} - 16z^{-4} \\
 & 6z^{-2} + 20z^{-3} + 16z^{-4} \\
 & \vdots
\end{array}$$

$\therefore$ $x[0] = 0$

$x[1] = 1$

$x[2] = 0$

$x[3] = -3$

$x[4] = -4$

$\vdots$

Inverse Z-Transform via PFE $\rightarrow$ If degree of $B(z) =$ degree of $A(z)$ can't use PFE on $X(z) = \frac{B(z)}{A(z)}$

$X(z) = \frac{B(z)}{A(z)}$ $\therefore$ (Note if degree $B(z) = A(z) \rightarrow X(z) = x[0] + \frac{R(z)}{A(z)}$)

$\downarrow$ o.w.

$$\frac{X(z)}{z} = \frac{B(z)}{zA(z)}$$

can perform PFE on either of these forms

12.02.03 Ch 11 Z-Transform / DTS.

Inverse Z-Transform $\frac{X(z)}{z}$

DISTINCT POLES poles $\{p_1, p_2, \ldots, p_N\}$ are distinct & non-zero

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{(z-p_1)} + \frac{c_2}{(z-p_2)} + \cdots + \frac{c_N}{(z-p_N)}$$

$$c_0 = \left[z \frac{X(z)}{z} \right]_{z=0} = X(0)$$

$$\{c_i\} = \left\{ \left[(z-p_i) \frac{X(z)}{z} \right] \Big|_{z=p_i} \right\} \qquad i = 1, 2, \ldots, N$$

$$X(z) = c_0 + \frac{c_1 z}{z-p_1} + \frac{c_2 z}{z-p_2} + \cdots + \frac{c_N z}{z-p_N}$$

$$x[n] = c_0 \delta[n] + c_1 p_1^n + c_2 p_2^n + \cdots + c_N p_N^n \qquad n = 0, 1, 2, \ldots$$

modes of convergence
(as in Laplace Transform)

complex poles occur in complex conjugate pairs s.t.

$c_1 p_1^n + \bar{c}_1 \bar{p}_1^n$ is a term that can be expressed as.

$$2|c_1| \sigma^n \cos(\Omega n + \measuredangle c_1)$$

(remember $p = \sigma e^{j\Omega}$)

$\sigma = |p|$

$\Omega = \measuredangle p$

12.02.03 Ch 11 - Z Transform / DTS.

REPEATED POLES. p_1 is repeated r times.

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \cdots + \frac{c_r}{(z-p_1)^r} + \frac{c_{r+1}}{z-p_{r+1}} + \cdots + \frac{c_N}{z-p_N}$$

$$c_r = \left[(z-p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$

$$c_{r-1} = \left[\frac{d}{dz}(z-p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$

$$\vdots$$

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{dz^i}(z-p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$

$$X(z) = c_0 + \frac{c_1 z}{z-p_1} + \frac{c_2 z}{(z-p_1)^2} + \cdots + \frac{c_r z}{(z-p_1)^r} + \frac{c_{r+1} z}{z-p_{r+1}} + \cdots + \frac{c_N z}{z-p_N}$$

from transform tables...

$$\frac{c_2 z}{(z-p_1)^2} \longrightarrow c_2 n p_1^{n-1} u[n]$$

$$\frac{c_3 z}{(z-p_1)^3} \longrightarrow \frac{1}{2} c_3 n(n-1) p_1^{n-2} u[n-1]$$

$$\vdots$$

$$\frac{c_i z}{(z-p_1)^i} \longrightarrow \frac{c_i}{(i-1)!} n(n-1)\cdots(n-i+2) p_1^{n-i+1} u[n-i+2]$$

In general, $x[n] \xrightarrow{n\to\infty} 0$ iff all poles are s.t. $|p_i|<1$ for $i=1,2,\ldots,N$

$x[n] \xrightarrow{n\to\infty} k$ constant iff $|p_i|<1$ except one of the $|p_i|=1$.

12.0203 Ch 11 Z-transform/DTS.

TRANSFER FN REPRESENTATION.

First-order case (LTI DTS)

$$y[n] + a y[n-1] = b x[n]$$

$$\downarrow \qquad \downarrow$$

$$Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z)$$

$$(1+az^{-1})Y(z) = -ay[-1] + bX(z)$$

$$Y(z) = -\frac{ay[-1]}{1+az^{-1}} + \frac{b}{1+az^{-1}}X(z)$$

$$= -\frac{ay[-1]z}{z+a} + \frac{bz}{z+a}X(z)$$

if $y[-1]=0$ (no initial energy at time $n=0$)

then $Y(z) = \frac{bz}{z+a} X(z)$

$$\uparrow \; H(z) = \frac{bz}{z+a} \qquad Y(z) = H(z)X(z)$$

Second-order case

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1]$$

$\downarrow$ if no initial energy, i.e., $y[-1] = y[-2] = 0$

$$Y(z) = \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2} X(z)$$

$$\uparrow \; H(z) = \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2}$$

12.03.03 Ch 11 Z Transform/DTS.

N^{th} Order Case

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{i=1}^{M} b_i x[n-i]$$

↓ assuming initial conditions = 0.

$$Y(z) = \underbrace{\frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}}_{H(z)} X(z)$$

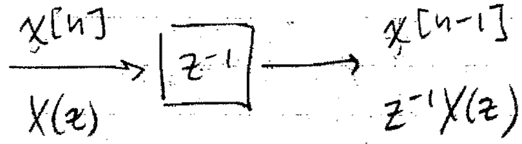
$H(z)$ $\quad\quad$ $Y(z) = H(z)X(z)$

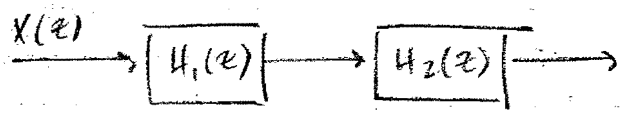
↓

$h[n]$ impulse response

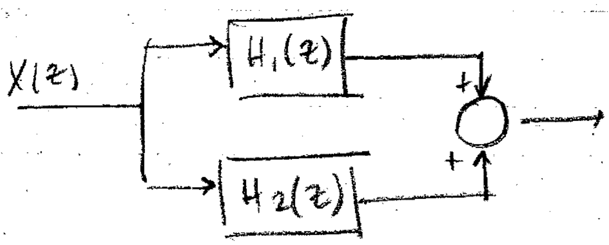
Transfer fn of Interconnections.

unit-delay element

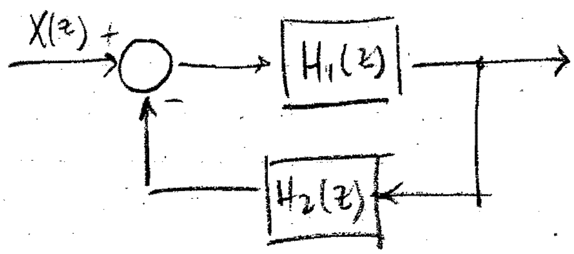




$Y(z) = H_1(z)H_2(z)X(z)$



$Y(z) = (H_1(z) + H_2(z))X(z)$



$$Y(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)} X(z)$$

12·03·03. Ch 11 - Z transform / DTS.

FREQUENCY RESPONSE

$$x[n] = C_1 \cos(\Omega_0 n) \quad n = 0, 1, 2, \ldots$$

$$\downarrow$$

$$X(z) = \frac{C_1 [z^2 - (\cos\Omega_0) z]}{z^2 - (2\cos\Omega_0) z + 1}$$

If $H(z) = B(z)/A(z)$ & no initial energy in system at $n=0$,

then

$$Y(z) = \frac{C_1 B(z) [z^2 - (\cos\Omega_0) z]}{A(z) [z^2 - (2\cos\Omega_0) z + 1]}$$

Note $H(z) = \frac{B(z)}{A(z)}$ ← assumed STABLE

$$= \frac{C_1 B(z) [z^2 - (\cos\Omega_0) z]}{A(z)(z - e^{j\Omega_0})(z - e^{-j\Omega_0})}$$

$$\therefore \frac{Y(z)}{z} = \frac{\eta(z)}{A(z)} + \frac{c}{z - e^{j\Omega_0}} + \frac{\bar{c}}{z - e^{-j\Omega_0}}$$

$$c = \frac{C_1}{2} H(e^{j\Omega_0})$$

$$\Downarrow$$

$$\bar{c} = \frac{C_1}{2} \overline{H(e^{j\Omega_0})}$$

$$Y(z) = \frac{z\eta(z)}{A(z)} + \frac{C_1}{2} H(e^{j\Omega_0}) \frac{z}{z - e^{j\Omega_0}} + \frac{C_1}{2} \overline{H(e^{j\Omega_0})} \frac{z}{z - e^{-j\Omega_0}}$$

$$y_{tr}[n] \xrightarrow{n \to \infty} 0$$ since $H(z)$ is assumed stable

$$y_{ss}[n] = C_1 |H(e^{j\Omega_0})| \cos[\Omega_0 n + \measuredangle H(e^{j\Omega_0})]$$

↑ steady-state output resulting from input $x[n]$

Since $H(z)$ is stable, $h[n]$ is absolutely summable

$$\therefore H(\Omega) = H(z)\big|_{z = e^{j\Omega}}$$

$$|H(e^{j\Omega_0})| = |H(\Omega)|_{\Omega = \Omega_0}$$

$$\measuredangle H(e^{j\Omega_0}) = \measuredangle H(\Omega)|_{\Omega = \Omega_0}$$

$$\therefore y_{ss}[n] = C_1 |H(\Omega_0)| \cos(\Omega_0 n + \measuredangle H(\Omega_0)) \Longleftarrow$$